Operations and Calculations

➢ Order of Operations

- **Review of PEDMAS**

When there is more than one operation in a mathematical expression, we must follow the correct *order of operations*:

PEDMAS

Parentheses
Exponents
*Divide or M*ultiply
Add or Subtract

- Carry out the operations *within each level* from *left-to-right*.
- Continue until all the operations are completed

Ex 1:

$$-5^2 + 4 - 2 \times (6 + 5) = ?$$

Answer:

Parentheses	$= -5^2 + 4 - 2(11)$
Exponents	$= -1 \times 25 + 4 - 2 \times (11)$
Multiply or Divide	$= -25 + 4 - 22$
Add or Subtract	$= -21 - 22$
Add or Subtract	$= -43$

The answer to the mathematical expression is -43.

The same order of operations applies to mathematical expressions involving variables (algebraic expressions).

Ex 2:

Given the algebraic expression $-2x^2 - \frac{8}{(x+1)}$,

a) List the order of operations.
b) Evaluate the expression when $x = -3$.

Answer: a)

Parentheses		Operate x add 1 to get $(x + 1)$
Exponents		Operate x^2
Multiply or Divide		Multiply -2 and x^2
Multiply or Divide		Divide 8 by $(x + 1)$
Add or Subtract		Subtract $\frac{8}{x+1}$ from $-2x^2$

b) Following the order of operations listed in part a), we evaluate the expression when $x = -3$:

$$
\begin{aligned}
-2x^2 - \frac{8}{x+1} &= -2(-3)^2 - \frac{8}{(-3+1)} \\
&= -2 \times (-3)^2 - \frac{8}{-2} \\
&= -2 \times 9 - \frac{8}{-2} \\
&= -18 - \frac{8}{-2} \\
&= -18 - (-4) \\
&= -18 + 4 \\
&= -14
\end{aligned}
$$

- **Reverse (Inverse) Operations**

In mathematics, an inverse operation is one that *reverses* the effect of another operation. Here are some examples:

1. Addition and Subtraction:

   - Addition: $a + b$
   - Inverse operation (subtraction): $a + b - b = a$

   Subtracting b reverses the effect of adding b to get back to a

2. Multiplication and Division:

   - Multiplication: $a \cdot b$
   - Inverse operation (division): $\frac{a \cdot b}{b} = a$

   Dividing by b reverses the effect of multiplying by b to get back to a

3. Exponents and Roots **:

   - Exponentiation: a^b
   - Inverse operation (root): $\sqrt[b]{a^b} = a$

   Taking b^{th} root reverses the effect of raising a to power b to get back to a

** Strictly speaking, $\sqrt[b]{a^b}$ returns $-a$ when a is negative and b is even. Listed below are a few examples to show this extra point:

$$(-2)^2 = 4 \text{ vs. } \sqrt[2]{4} = \sqrt{4} = 2$$

$$(-2)^3 = -8 \text{ vs. } \sqrt[3]{-8} = -2$$

$$(-3)^5 = -243 \text{ vs. } \sqrt[5]{-243} = -3$$

$$(-3)^6 = 729 \text{ vs. } \sqrt[6]{729} = 3$$

In the example of $(-3)^6 = 729$, to get back to the value of -3, we write $-\sqrt[6]{729} = -3$ by convention.

Ex 3: Given the algebraic equation: $-\frac{x^2+1}{4-2} = -7$.

a) List the order of operations on the left-hand side to arrive at the answer of -7 on the right.
b) Solve the equation for x by reversing the operations.

Answer:

a) Note that in algebraic equations there is always an <u>implied</u> parentheses in the parts of expressions involving fractions. In our current example, the equation really means:

$$-\frac{(x^2+1)}{(4-2)} = -7$$

<u>P</u>arentheses (Numerator)	Operate x^2 then add 1 to get (x^2+1)
<u>P</u>arentheses (Denominator)	Subtract 2 from 4 to get 2
<u>M</u>ultiply or <u>D</u>ivide	Divide (x^2+1) by 2
<u>M</u>ultiply or <u>D</u>ivide	Multiply $\frac{x^2+1}{2}$ by -1 to get -7

b) Reversing the operations and going backwards, we undo each operation to get back to input x:

1) Divide -7 by -1 to get the value of $\frac{x^2+1}{2}$:

$$\frac{x^2+1}{2} = 7$$

2) Multiply the result by 2 to get back to the value of x^2+1:

$$x^2 + 1 = 14$$

3) Subtract 1 from the result to get back to the value of x^2:

$$x^2 = 13$$

4) Since the power of x is even (2), take both positive and negative square root of 13 to get back to the value of x:

$$x = \pm\sqrt{13}$$

The two solutions of the equation are $x = \sqrt{13}$ and $x = -\sqrt{13}$.

- **Number Bases**

- **Base 10**

It is widely believed that the use of base 10 (*decimal system*) in many cultures is due to humans having 10 fingers. This theory is based on anthropological and historical evidence showing that ancient humans used their fingers for counting, which naturally led to a base 10 system. Base 10 system uses 10 digits (0, 1, 2, 3, …, 8, 9) and its place values are:

	Thousands	Hundreds	Tens	Ones	Tenth	Hundredth	Thousandths	
…	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	…
…	1000	100	10	1	0.1	0.01	0.001	…

Note: 10^{-n} *is equivalent to* $\frac{1}{10^n}$*. For example,* $10^{-3} = \frac{1}{10^3} = 0.001$.

When we have a number 6791 in base 10, it is equal to $6 \times 10^3 + 7 \times 10^2 + 9 \times 10 + 1 \times 10^0$.

However, there are other number base systems that more suitable for specific types of applications.

Here are some examples:

- ✓ Base 2, or *binary*, is used for computers because it aligns perfectly with the digital nature of electronic circuits, which have two states: on and off, or 1 and 0. This simplicity makes binary the foundation of all computing and digital communications.

- ✓ Base 6, or *senary*, is also useful because hexagonal shapes are fundamental in nature and efficient for tiling and packing. For example, honeycombs use a hexagonal structure, and some cultures historically used base 6 for counting. Additionally, base 6 is practical because it relates well to the number of days in a year. The number 365 is approximately divisible by 6, making it a convenient system for certain calendrical and timekeeping purposes.

- ✓ Base 16, or *hexadecimal*, is another important base, especially in computing. It simplifies the representation of binary-coded values by grouping binary digits into sets of four (as $2^4 = 16$), making it easier to read and write large binary numbers.

- ✓ Base 60, or the *sexagesimal* system, was developed by the Sumerians and Babylonians around 3000 BCE. It is known for its extensive divisibility (factors of 60: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60), making it highly practical for complex calculations, particularly in astronomy and geometry. The influence of the base 60 system persists today in our timekeeping (60 seconds in a minute, 60 minutes in an hour) and in the measurement of angles (360 degrees in a circle). The sexagesimal system's ease of division and its suitability for astronomical calculations and time keeping made it a robust choice for the Sumerians and Babylonians. While base 60 is excellent for division and calculations involving fractions, base 10 is simpler for basic arithmetic and everyday use. Most day-to-day transactions and record-keeping do not require the complex divisibility offered by base 60.

In a base-n number system, there are n available digits before we need to carry over. For example, in base 6, the digits are 0, 1, 2, 3, 4, 5. Once we reach the count of 6, we carry over and represent the count of 6 with 10. Note that this does not represent the count of 10 as in base 10. To avoid confusion, we write it as $10_{(6)}$ to be clear that this is in base 6 and represents the value of 6.

Similarly, in base 16, the digits are:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F

Here's a breakdown:

- 0-9**:** These digits represent the same values as in the decimal system (0 through 9).
- A-F: These letters represent the values 10 through 15 as in the decimal system.

Ex 4: What is the value of $1C3_{16}$ in the decimal system?

Answer:

The hexadecimal (base 16) number 1C3 translates to $1 \times 16^2 + C \times 16^1 + 3 \times 16^0$.

In decimal, this is equal to $1 \times 256 + 12 \times 16 + 3 = 256 + 192 + 3 =$451.

Ex 5: How should we represent the number 255 in hexadecimal system?

Answer:

Divide 255 by 16: the quotient is 15 (F in hex), and the remainder is 15 (F in hex).

So, 255 in decimal is FF in hexadecimal.