Ratio, Rate, and Proportion

- **Basic Concepts**

- **Ratio**

A **ratio** is a way to show how much of one thing there is compared to another. It tells us how many times one number can fit into another. For example, if the ratio of apples to oranges in a basket is 7 to 11, this means for every 7 apples, there are 11 oranges. Ratios can be expressed in several forms:

- As a fraction: the ratio of 7 to 11 can be written as $\frac{7}{11}$.
- With a colon: using the same numbers, it can be expressed as $7{:}\ 11$.

To compare ratios, write them as fractions. The ratios are equal if they are equal when written as fractions.

Example 1： Are the ratios 5 to 20 and 4 to 16 equivalent?

Answer:

The ratios are equal if 5/20 = 4/16.

These are equal if their cross products are equal; that is, if 5 × 16 = 4 × 20. Since both products equal 80, the answer is yes, the ratios are equal. Another way of seeing that the two ratios are equivalent is that both ratios can be reduced to 1/4.

- **Rate**

Both ratios and rates involve comparing quantities, but there is a slight difference. A ratio is a comparison between two quantities of the same unit or type. For example, comparing the number of apples to oranges as 7 to 11. A **rate** is a special kind of ratio where the two quantities have different units. For example, speed is a rate because it compares distance to time, like miles per hour.

Example 2: A phone company charges $0.84 for 7 minutes of long distance, and a student reads 10 pages in 8 minutes.

Answer:

The first rate: $\frac{\$0.84}{7\text{ min}} = \frac{\$0.12}{1\text{ min}}$

The second rate: $\frac{10\text{ pages}}{8\text{ min}} = \frac{5\text{ pages}}{4\text{ min}}$

The first rate is called a unit rate. In a unit rate, the denominator quantity is 1. A unit rate is often used for comparing the cost of two similar options.

Example 3: If a 12-ounce box of cereal sells for \$2.40, and a 16-ounce box sells for \$2.88, which is the better buy?

Answer:

The unit rate of the first box: $\frac{\$0.24}{12} = \$0.20.$

The unit rate of the second box: $\frac{\$2.88}{16} = \$0.18.$

Therefore, the second box is a better buy.

- **Proportion**

A proportion is an equation with a ratio on each side. It is a statement that two ratios are equal.

7/11 = 14/22 is an example of a proportion.

When one of the four numbers in a proportion is unknown, cross products may be used to find the unknown number. This is called solving the proportion. Question marks or letters are frequently used in place of the unknown number.

Example 4: Suppose a person drives 126 miles in 3 hours. At the same speed, how many miles would the driver travel in 4 hours? Because the rate of travel (speed) remains the same, a proportion can be written.

Answer:

The unknown quantity, the distance traveled by the car in 4 hours, can be indicated by x. Therefore, the two ratios 126 / 3 and x / 4 form a proportion.

$$\frac{126 \text{ miles}}{3 \text{ hours}} = \frac{x}{4 \text{ hours}}$$

Multiplying both sides by 4, yields x = 168 miles:

$$\frac{126}{3} \cdot 4 = \frac{x}{4} \cdot 4$$

$$42 \cdot 4 = x$$
$$x = 168$$

Or, using cross multiplication, also yields the same value for x:

$$126 \cdot 4 = 3x$$
$$x = \frac{126 \cdot 4}{3}$$
$$x = 42 \cdot 4$$
$$x = 168$$

At the same speed, the driver would travel 168 miles in 4 hours

Summarizing the Concepts

A ratio compares the magnitude of two quantities. When the quantities have different units, then a ratio is called a rate. A proportion is a statement of equality between two ratios.

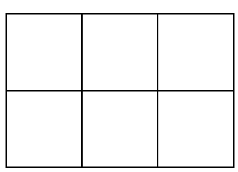
- **Ratio and Proportions in Geometry**

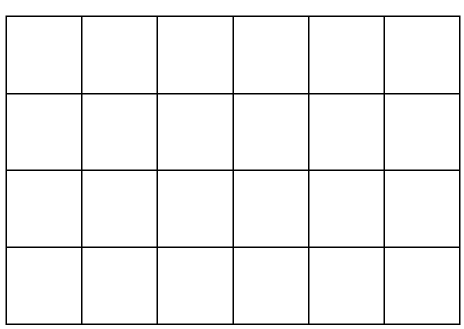
- **Ratios in Geometry:**

A ratio in geometry might compare lengths, angles, areas, or any other measurable attributes. For instance, in a rectangle, the ratio of the length to the width can describe the shape's proportions. If a rectangle has a length of 10 units and a width of 5 units, the ratio of length to width is 10:5, which simplifies to 2:1.

- **Proportions in Geometry:**

As mentioned previously, a proportion is an equation that states that two ratios are equal. It's a way to express that two or more ratios have the same relationship. For example, if one rectangle has a length to width ratio of 3:2, and another rectangle also has its dimensions in the ratio of 6:3, these two ratios form a proportion because 3:2 is equivalent to 6:4. In simpler terms, when two shapes have the same proportions, it tells us that the two shapes are "in scale" with each other. They look the same in terms of their shape but might differ in size. This **similarity** in appearance, regardless of size, is what we mean when we say that they are “in scale” or proportional. Each part of one shape corresponds evenly with each part of the other shape, just on a different scale.





- **Applications of Ratios and Proportions in Geometry:**

Similar Figures: When two geometric shapes are similar, all corresponding linear dimensions are proportional. This means that corresponding sides, heights, and diagonals of these figures maintain the same ratio.

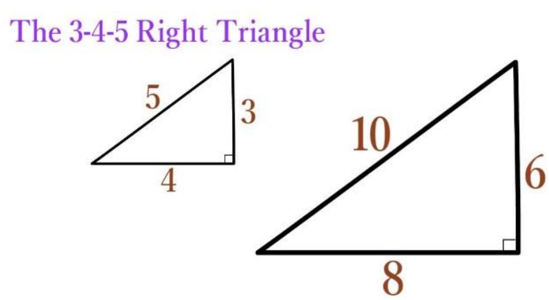
Example 5: Two triangles, Triangle A and Triangle B, are similar. The sides of Triangle A are 6 cm, 8 cm, and 10 cm. The shortest side of Triangle B is 9 cm.

a) Find the perimeter of Triangle B.
b) Show that the two triangles are right angled and determine their hypotenuse.

Answer:

a) Since the two triangles are similar, the sides of Triangle B are proportional to the sides of Triangle A. The shortest side of Triangle B (9cm) corresponds to the shortest side of Triangle A (6cm) and the ratio is: $\frac{9}{6} = \frac{3}{2}$. This means that Triangle B is 1.5 times larger than Triangle A is all linear dimensions (sides, heights, and diagonals). Since each side is 1.5 times larger, we can conclude that the perimeter is also 1.5 times larger: $(6 + 8 + 10) \cdot \frac{3}{2} = 24 \cdot \frac{3}{2} = 36$.
The perimeter of Triangle B is 36.

b) Since $6: 8: 10 = 3: 4: 5$ and $3^2 + 4^2 = 5^2$, we notice that Triangle A and therefore Triangle B both have the same shape as the 3-4-5 right triangle. The hypotenuse of Triangle A is 10, and correspondingly, the hypotenuse of Triangle B is $10 \times 1.5 = 15$.



Scale Models: Ratios are critical when creating scale models of objects like buildings or bridges. The model's dimensions are proportional to the actual object's dimensions, allowing architects and engineers to study and showcase their designs at a manageable scale.

Calculating Areas and Volumes: Ratios and proportions can help calculate unknown areas or volumes when similar figures are involved. For instance, if the side lengths of one square are twice those of another, its area is four times greater, because the area of a square is proportional to the square of its side length.

Understanding and applying ratios and proportions in geometry allows for accurate design, construction, and interpretation of spatial relationships, making these concepts critical tools in many practical and theoretical applications.

Example 6: An architect is creating a scale model of a new office building. The actual building will be 180 meters tall, and the architect decides to use a scale of 1:1000 for the model.

a) What will be the height of the scale model?
b) If the actual width and length of the building are 40 meters and 50 meters, what will be the width and length of the model?
c) What would be the volumes of the model and the actual building? Compare the ratio in the volumes.
d) What would be the space diagonal of the actual building?

Answer:

a) The height of the scale model should be 1000 times smaller:

$\frac{180\text{ m}}{1000} = 0.18\text{ m} = 18\text{ cm}.$

b) The width and length of the scale model should also be in the same scale:

$\frac{40\text{ m}}{1000} = 0.04\text{ m} = 4\text{ cm}$ and $\frac{50\text{ m}}{1000} = 0.05\text{ m} = 5\text{ cm}.$

c) The volume of the model:

$18\text{ cm} \times 4\text{ cm} \times 5\text{ cm} = 360\text{ cm}^3$

The volume of the actual building:

$180\text{ m} \times 40\text{ m} \times 50\text{ m} = 360\ 000\text{ m}^3$ which is equivalent to:

$$\begin{aligned}360\ 000\text{ m}^3 &= 360\ 000\ (100\text{cm})^3 \\ &= 360\ 000 \times 1\ 000\ 000 \\ &= 360 \times 1\ 000\ 000\ 000\text{ cm}^3\end{aligned}$$

Therefore, the ratio in the volumes is $1{:}\,10^9$, which is the cube of the ratio of linear dimensions:

$$(1{:}\,1000)^3 = 1{:}\,10^9.$$

d) We can first find the space diagonal of the model, then apply the ratio of linear dimensions to find the space diagonal of the building:

$$\begin{aligned}\text{The space diagonal of the model} &= \sqrt{18^2 + 4^2 + 5^2} \\ &= \sqrt{324 + 16 + 25} \\ &= \sqrt{365}\text{ cm}\end{aligned}$$

Therefore, the space diagonal of the building is $1000 \times \sqrt{365}$ cm or $10\sqrt{365}$ m.