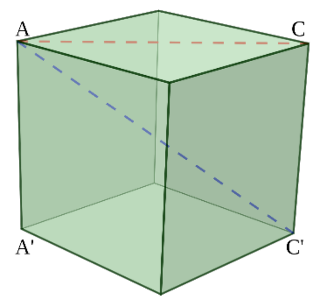
3D Geometry

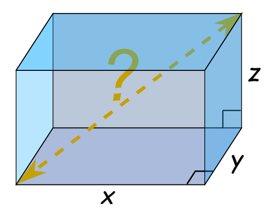
- **3-Dimensional Pythagorean Theorem**

In geometry, a rectangular prism is a three-dimensional shape with six rectangular faces. When discussing the *diagonals* of a rectangular prism, we differentiate between the **face diagonal** and the **space diagonal**, each representing different measurements within the prism:



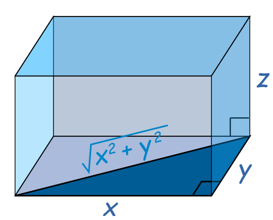
1. **Face Diagonal**: This refers to the diagonal line connecting two opposite corners of any rectangular face of the prism. In the diagram to the right, line segment AC is a face diagonal. Since each face of the prism is a rectangle, the face diagonal divides the rectangle into two right triangles. It is longer than any of the sides of the rectangle but shorter than the space diagonal of the prism. The length of the face diagonal can be found using the Pythagorean theorem applied to the lengths of two adjacent sides of the rectangle.

2. **Space Diagonal**: This is the diagonal line that stretches from one *vertex* (corner) of the prism to the opposite vertex, passing through the interior of the prism. It connects the furthest points in the prism, making it the *longest diagonal* within the shape. In the diagram to the right, line segment AC' is a space diagonal. The space diagonal spans across three dimensions — length, width, and height — and its length can also be calculated using the Pythagorean theorem, but in a three-dimensional context, incorporating all three dimensions of the prism.



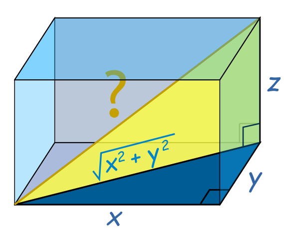
In summary, while the face diagonal is confined to the plane of a face and spans two dimensions, the space diagonal traverses the entire volume of the prism, touching on all three dimensions.

Let's calculate the distance from the bottom-most left front corner to the top-most right back corner of a rectangular prism using the 3-dimensional Pythagorean theorem.



First, consider the triangle on the bottom face of the rectangular prism. According to the Pythagorean theorem, the diagonal c across this face is calculated as:

$c^2 = x^2 + y^2$, where x and y represent the length and width of the rectangular prism.



Next, form a right triangle where this diagonal $c = \sqrt{x^2 + y^2}$ serves as the base, and the height of the rectangular prism z is the other leg. Applying the Pythagorean theorem again to this setup, we calculate the diagonal from the bottom-most left front corner to the top-most right back corner, which is the space diagonal of the solid. The following shows the steps of deriving the formula.

Using the sides $c = \sqrt{x^2 + y^2}$ and z, the formula for the space diagonal d is:

$$d = \sqrt{\left(\sqrt{x^2 + y^2}\right)^2 + z^2}$$

Simplifying this, we get:

$$d = \sqrt{x^2 + y^2 + z^2}.$$

This gives us the result for the distance from one corner of the rectangular box to its opposite corner through the interior. In other words, the formula calculates the length of the *space diagonal*.

Example 1: Determine the shortest distance from one vertex of a $4\text{cm} \times 6\text{cm} \times 8\text{cm}$ rectangular prism to the opposite vertex through the interior of the prism.

Answer:

The shortest distance from one vertex of a $4\text{cm} \times 6\text{cm} \times 8\text{cm}$ rectangular prism to the opposite vertex through the interior of the prism is essentially the length of the space diagonal d:

$$d = \sqrt{4^2 + 6^2 + 8^2}$$

$$d = \sqrt{16 + 36 + 64}$$

$$d = \sqrt{116}$$

$$d = \sqrt{4 \cdot 29}$$

$$d = 2\sqrt{29}$$

The shortest distance is $2\sqrt{29}\ \text{cm}$ in exact values, or about 10.77cm when rounded to the hundredth.

- **Net Diagrams of a 3-Dimensional Shape**

To fully grasp how to calculate the surface area of three-dimensional solids, it's essential to familiarize oneself with net diagrams of these shapes. Net diagrams are two-dimensional representations that can be folded to form the corresponding three-dimensional solid. These diagrams effectively "unfold" the solid, laying out all its faces flat, which allows us to see and measure each face individually.

Understanding net diagrams is crucial because they simplify the process of calculating surface area. By displaying all faces of a solid in a single plane, net diagrams allow us to easily apply area formulas to each separate face and then sum these areas to find the total surface area of the solid.

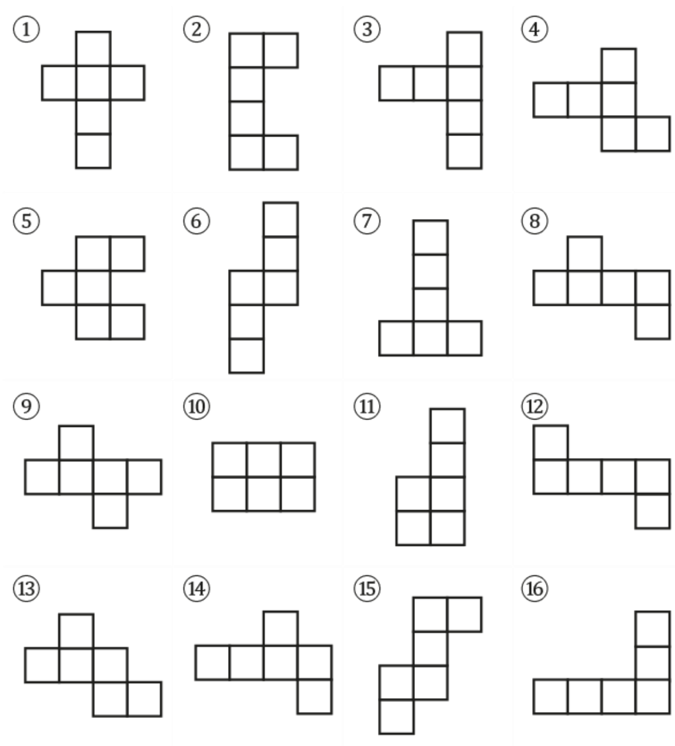
Imagine you're opening a cardboard cube and flattening it out on the ground. This flat arrangement is essentially what a net diagram of the cube looks like. A net diagram is a two-dimensional layout that shows all the faces of the cube spread out flat without any overlaps.

When you unfold the cube, you expose all six faces—each one a square. These squares are represented in the net diagram as separate squares connected at the edges. By arranging these squares in a specific way, you can see how they fit together to form the three-dimensional shape when folded up.

For many three-dimensional shapes, including cubes, there are multiple ways to draw a net diagram. Each net represents a different arrangement of the shape's faces when laid out flat, and choosing among them can depend on the application or the easiest visualization for understanding.

Example 2: The net diagrams of cubes are some of the simplest to visualise and it is a fun practice of your spatial skills to see how many you can create. In fact, there are **11 distinct net diagrams that make a cube**.

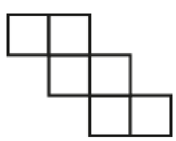
The diagram on the right shows 16 different arrangements of 6 squares that all look like they could be cube nets, but 6 of them are not. Can you work out which are valid nets of a cube?



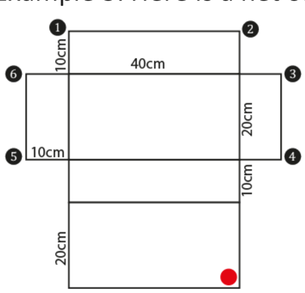
Answer:

The ten valid nets of a cube are 1, 4, 6, 7, 8, 9, 12, 13, 14 and 15, while nets 2, 3, 5, 10, 11 and 16 cannot make a cube and they are the non-nets.

There is one valid net of a cube that's missing:



Example 3: Here is a net of a rectangular prism with side lengths 10cm, 20cm and 40cm:



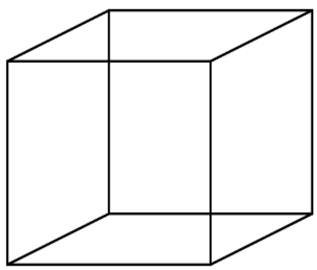
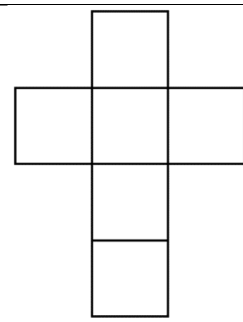
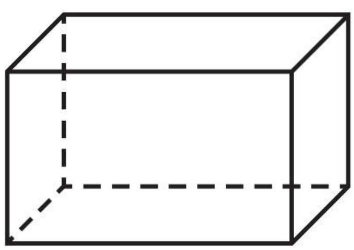
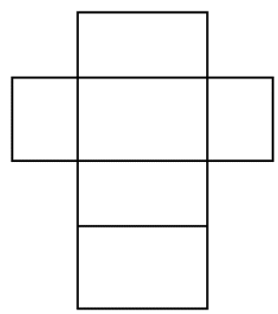
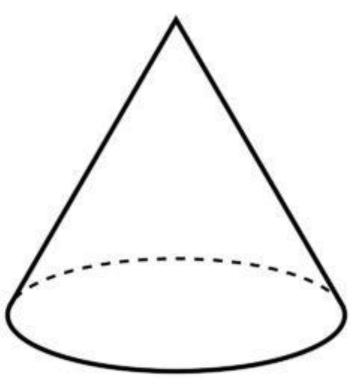
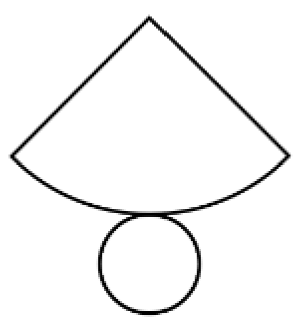
Look for the vertex (corner) marked with the red dot. Using your spatial skills again, can you work out which other vertices, labelled 1 – 6, will join up with the red dot, when the rectangular prism is in its 3D form?

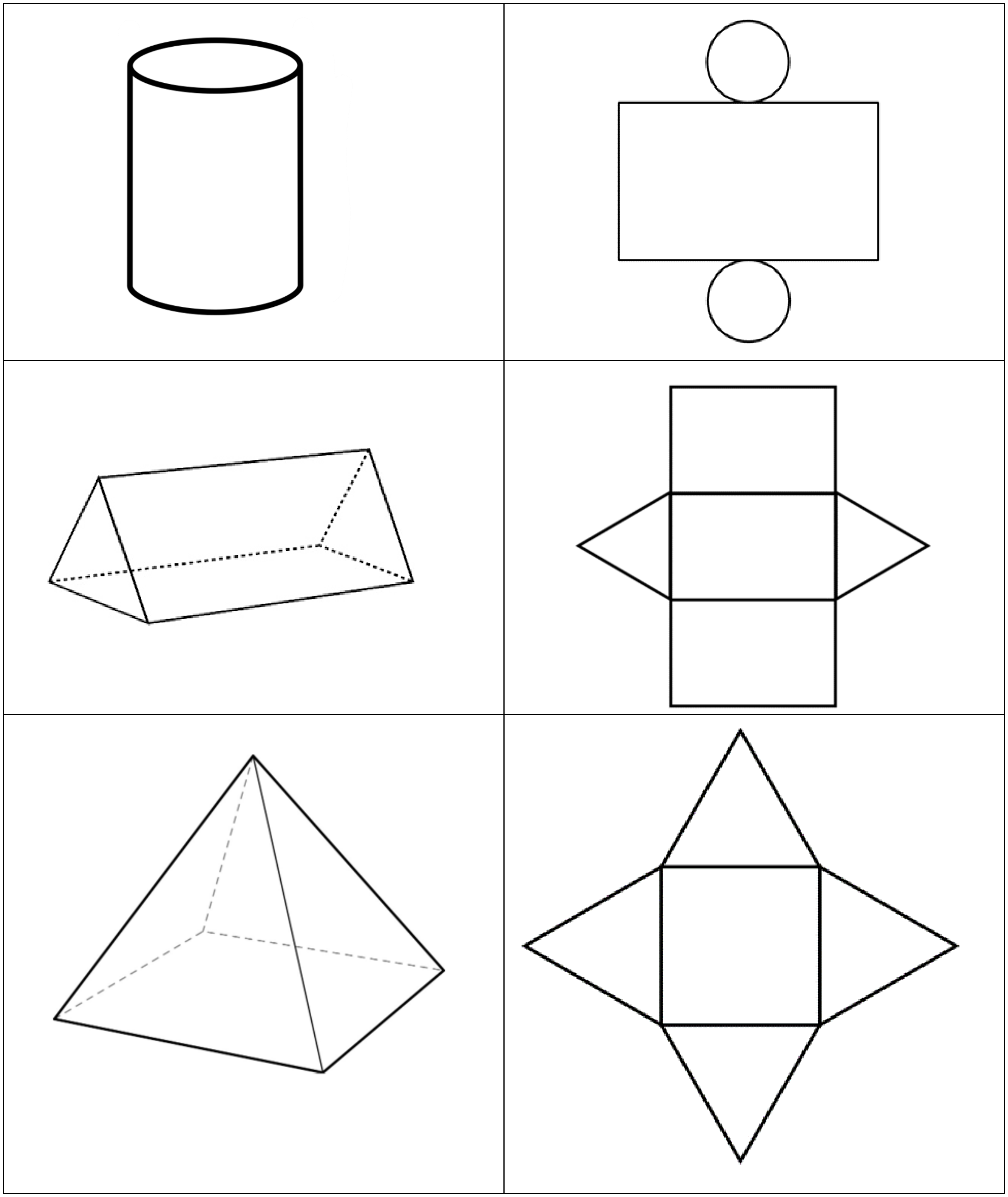
(When you're ready, scroll to the next page to reveal the answer.)

Answer:

Vertices 2 and 3 meet the vertex marked with the red dot.

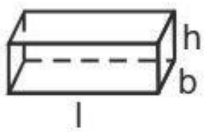
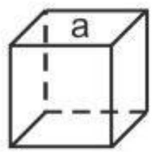
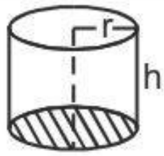
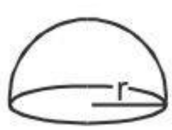
Here is a summary of some common 3-dimensional solids along with one of their possible net diagrams:

3D Solids	**Nets**
	
	
	



Net diagrams are particularly helpful in educational settings because they provide a visual understanding of how different shapes compose the surface of a solid. This visualization aids in comprehending more complex shapes and in appreciating how surface area is a measure of all the exposed material that would be needed to cover a 3D object completely, making the concept more intuitive and relatable.

- **Formulae for the Volumes and the Surface Areas of Common 3-Dimensional Solids**

Name	Figure	Curved Surface area	Total surface area	Volume
Cuboid		$2h(l+b)$	$2(lb+bh+lh)$	lbh
Cube		$4a^2$	$6a^2$	a^3
Right circular cylinder		$2\pi rh$	$2\pi r(r+h)$	πr^2h
Right circular cone		πrl	$\pi r(l+r)$	$\frac{1}{3}\pi r^2h$
Sphere		–	$4\pi r^2$	$\left(\frac{4}{3}\right)\pi r^3$
Hemi-sphere		$2\pi r^2$	$3\pi r^2$	$\left(\frac{2}{3}\right)\pi r^3$

Note: "Cuboid" is another name for a rectangular prism and "curved surface area" is also referred to as the lateral surface area (the faces minus the base and/or the top face).