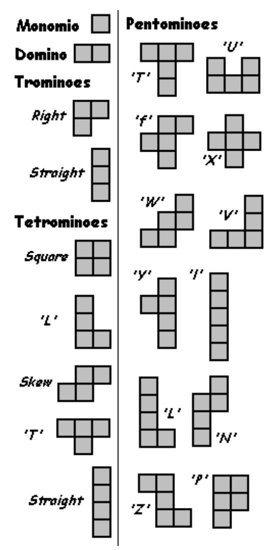
Patterns

➢ Polyomino Tiling

Polyomino tiling is a mathematical and recreational problem involving the arrangement of polyominoes to cover a given area, such as a board or a plane, *without overlapping and without leaving any gaps*. Polyominoes are shapes formed by joining one or more equal squares edge to edge. They are essentially two-dimensional geometric figures made up of joined unit squares, and they can be highly varied in shape depending on the number of squares and how they are arranged.

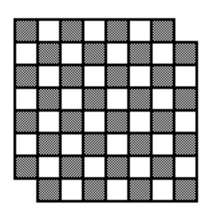


- **Polyomino Types**: The simplest type of polyomino is the "monomino," which is a single square. As more squares are added, different shapes are formed:
  - **Domino**: Made up of two squares.
  - **Tromino**: Made up of three squares.
  - **Tetromino**: Made up of four squares (famous from the game Tetris).
  - **Pentomino**: Made up of five squares.
  - More complex forms include hexominoes (six squares), heptominoes (seven squares), etc.
- **Tiling Rules**: The primary rule in polyomino tiling is that the polyominoes must cover the target area completely without any overlaps or gaps. They are usually allowed to be rotated or flipped.

Polyomino tiling captures interest for both its mathematical depth and its applications. It is related to topics in combinatorics and geometry. The problem has practical implications in areas such as materials science (e.g., tiling surfaces with tiles of different shapes), computer science (e.g., data storage and retrieval), and more broadly in the field of design and architecture.

A variety of intriguing mathematical challenges stem from attempts to completely cover the squares of an 8 by 8 checkerboard using different polyomino shapes. The simplest cases involve monominoes (single squares) and dominoes (two connected squares), where full coverage is straightforwardly achievable. However, the scenario becomes more complex and intriguing when modifications are made to the board.

One well-known puzzle asks whether it is possible to cover an 8 by 8 checkerboard with its opposite corners removed using dominoes. At first glance, this might seem feasible, but a logical approach called the "parity test" demonstrates that it is impossible. This test involves observing that each domino covers one square of each color on a traditionally alternately colored checkerboard. With opposite corners removed (which are the same color), the board ends up with an unequal count of dark and light squares—making domino tiling impossible, as there would always be leftover squares of one color.



Parity test coloring for the domino

- **parity** refers to whether a number is odd or even. In the context of the *parity test*, this concept is used to analyze the balance or imbalance in a system, such as the number of dark and light squares.

The concept of parity can also be used in the context of probability. Here is an example involving coin flipping.

Example 1: You are flipping a fair coin (equal chances of heads or tails), and you are interested in the probability of ending up with an even number of heads after a certain number of flips. Let's analyze the parity of the number of heads after three flips.

After three flips, there are 8 (2^3) possible outcomes. Here's how parity comes into play:

- **Even Parity**: an even number of heads can be achieved in the following ways:
  - ✓ 0 heads (TTT) - 1 outcome
  - ✓ 2 heads (HHT, HTH, THH) - 3 outcomes
- **Odd Parity**: An odd number of heads can be achieved in the following ways:
  - 1 head (HTT, THT, TTH) - 3 outcomes
  - 3 heads (HHH) - 1 outcome

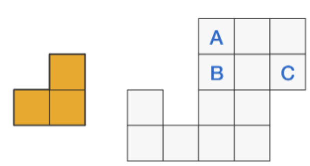
The total number of outcomes with an even number of heads is $1 + 3 = 4$, and the total number of outcomes with an odd number of heads is $3 + 1 = 4$.

In this scenario, the probability of ending up with an even number of heads after three flips is $\frac{4}{8} = 0.5$, and similarly, the probability of ending up with an odd number of heads is also 0.5.

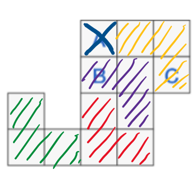
The concept of even or odd parity from flipping a fair coin does not change, regardless of how many times you flip the coin. The symmetry arises from the fact that a fair coin has the same chance to land on heads as it does on tails. Over a large number of flipping sequences, half the time you add a head (increasing the count by one and switching the parity), and half the time you add a tail (keeping the count and the parity the same).

Let's go back to the problems of polyomino tiling. Remember, to be considered a tiling, the polyominos need to cover the entire shape without any gaps or overlaps.

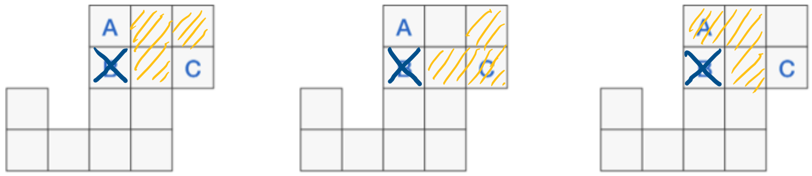
Example 2: If one of the squares marked with a letter is removed, the shape on the right can be tiled by the right tromino (L-shaped) on the left. Which square should be removed?



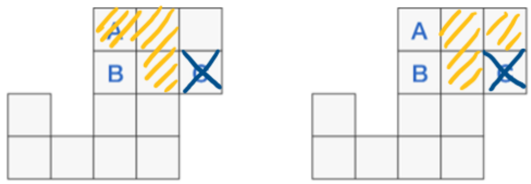
Only A, a tiling that excludes the square marked A is this:



If B is removed, at least 2 squares cannot be reached by the tromino as shown below.

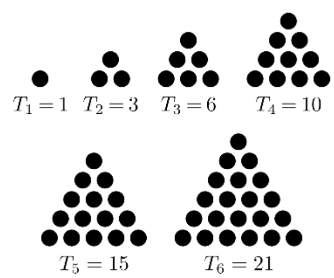


Similarly, if C is removed, at least 1 square cannot be reached by the tromino as shown below.



- **Triangular Numbers**

Triangular numbers are a captivating sequence in mathematics, where each number represents the total dots that can form an *equilateral triangle*. The sequence begins with 1, which is simply a single dot. Adding a second row with two dots gives us the second triangular number, 3. Continuing in this manner—by adding successive rows, each containing one more dot than the last—creates the triangular numbers.



If we let T_n represent the n^{th} triangular number, it is not hard to see that the n^{th} triangular number is:

$$T_n = T_{n-1} + n, \text{ where } T_1 = 1 \qquad (1)$$

or

$$T_n = 1 + 2 + 3 + 4 + 5 + 6 + \cdots + (n-1) + n \qquad (2)$$

Formula (1) is called a recursive formula. We can get to the value of the n^{th} triangular number by first getting to the $(n-1)^{\text{st}}$ triangular number. For example, since we know the 4^{th} triangular number, T_4, is 10, therefore, $T_5 = T_4 + 5 = 15$ is the 5^{th} triangular number.

Formula (2) is known as the general term (also referred to as the n^{th} term). It is a formula in terms of n which allows us to determine the n^{th} triangular number directly. For example, $T_5 = 1 + 2 + 3 + 4 + 5 = 15$ gives the value of the 5^{th} triangular number.

Notice that the n^{th} term formula for the triangular number, $T_n = 1 + 2 + 3 + 4 + \cdots + (n-1) + n$, is one example of an **arithmetic** sum.

- **Arithmetic Sequence and Sum**

An **arithmetic sequence** is a list of numbers in which the difference between any two consecutive terms is constant. This constant difference is often referred to as the "common difference," denoted as d.

- Components of an Arithmetic Sequence

  ✓ **Initial Term (T_1)**: This is the first term of the sequence.
  ✓ **Common Difference (d)**: This is the fixed amount that each term increases (or decreases) by, relative to the previous term.
  ✓ **General Term:** The n^{th} term of an arithmetic sequence can be expressed using the formula:

$$T_n = T_1 + (n-1)d$$

Example 3: Consider the sequence $2, 5, 8, 11, 14, \ldots$. Determine the 60^{th} term in the sequence.

The first term T_1 is 2.
The common difference (d) is $5 - 2 = 8 - 5 = 1 - 8 = 14 - 11 = 3$.

To find the 60^{th} term of this sequence, you can use the formula:

$$\begin{aligned} T_{60} &= 2 + (60-1) \cdot 3 \\ &= 2 + 59 \cdot 3 \\ &= 179. \end{aligned}$$

As the first term is 2 and the common difference is 3, it makes sense to add 3 fifty-nine times to get to the 60^{th} term.

An **arithmetic sum** refers to the total sum of terms in an arithmetic sequence. To calculate the sum of an arithmetic sequence, you can use the formula:

$$S_n = T_1 + T_2 + T_3 + \cdots + T_{n-1} + T_n,$$

$$S_n = \frac{(T_1 + T_2)\cdot n}{2}, \text{ where,}$$

- S_n is the sum of the first n terms of the sequence.
- n is the number of terms to be added.
- T_1 is the first term of the sequence.
- T_n is the last term of the sequence.

Example 4: Consider the sequence $2, 5, 8, 11, 14, \ldots$ from Example 3. Determine the sum of the first 60 terms in the sequence.

From Example 3, we found that the 60^{th} term is 179, and therefore, we are looking to find the sum of:

$S_{60} = 2 + 5 + 8 + 11 + \cdots + 176 + 179$

If we write the sum, but backwards, we have:

$S_{60} = 179 + 176 + 173 + 170 + \cdots + 5 + 2$

Add the two, we get:

$2 \cdot S_{60} = 181 + 181 + 181 + \cdots + 181$

$2 \cdot S_{60} = 60 \cdot 181$

$S_{60} = 30 \cdot 181$

$S_{60} = 5430.$

This is essentially following what the formula was saying:

$$S_{60} = \frac{60 \cdot (T_1 + T_{60})}{2} = \frac{60 \cdot (2 + 179)}{2} = \frac{60 \cdot 181}{2} = 5430.$$

Now that we understand how to find the sum of terms of an arithmetic sequence, we can simplify the general term for the n^{th} triangular number:

$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{(1+n) \cdot n}{2}.$$

Example 5: Determine the value of the 20^{th} triangular number.

$\frac{(20+1) \cdot 20}{2} = 21 \cdot 10 = 210$, the 20^{th} triangular number has a value of 210.