Number Theory

1. Natural numbers

Natural numbers are numbers used for counting, such as 1, 2, 3, and 139.

Natural numbers can also be seen as positive integers.

Are natural numbers the same as whole numbers?

No. Natural numbers start from 1, and whole numbers start from 0.

In contest questions, we commonly see the word "natural number" in the question, and students can interpret it as positive whole numbers, or positive integers.

2. Prime numbers

$10 \div 1 = 10$
$10 \div 2 = 5$
$10 \div 5 = 2$
$10 \div 10 = 1$

In the above example, 10 can be divided by 1, 2, 5, and 10 without leaving remainders. Therefore, 1, 2, 5, 10 are *factors* of 10. In other words, 10 has four factors. On the other hand, 5 has only two factors: 1 and 5.

Prime numbers are those that have exactly two factors, such as 5.
Composite numbers are those with more than two factors, such as 10.

Special case: 1 has only one factor (itself!), so it is neither prime nor composite.

Number	**Factors of the number**	**Number of factors** *(count the second column)*	**Prime or composite**
2	1, 2	2	Prime
5	1, 5	2	Prime
15	1, 3, 5, 15	4	Composite
20	1, 2, 4, 5, 10, 20	6	Composite
29	1, 29	2	Prime
64	1, 2, 4, 8, 16, 32, 64	7	Composite
91	1, 7, 13, 91	4	Composite

Prime number fun fact:

Two is the only even prime number there is. Because all of other even numbers have at least 1, itself, and 2 as factors.

3. Palindrome

Palindrome is a word or a number that reads the same forward or backwards.

Name example:

Ada, Ava, Elle, Hannah, Otto

Number example:

2002, 1234321, 99099

Do you know anyone with a Palindrome name?

Do you think you will experience a palindrome year in your life time?

4. Factorials

Factorials is a mathematical operation; it is the product of all the positive integers less than or equal to the number specified.

Example:

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$100! = 100 \times 99 \times 98 \times \ldots \times 3 \times 2 \times 1$$

Let n be any natural number.

$$n = n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \ldots \cdot 3 \cdot 2 \cdot 1$$

Students typically see factorials used in permutation and combination questions. For example, how many ways are there to rearrange 6 penguins to sit in a line? The answer would be 6! or 720.

5. Divisibility rules

Is 6210315 divisible by 3 and 5? What if you need to solve it within 10 seconds?

Divisibility rules are the shortcuts that can help us decide whether a number is divisible by a single digit number, without performing the actual long division.

Divisibility rules come handy in many scenarios, such as prime factorization, whether a fraction can be further reduced, and etc.

Factor	*Property required from the dividend*	*example*
2	The last digit is even.	247584<u>6</u>
3	Sum of digit is a multiple of 3.	111111 1+1+1+1+1+1 = 6
4	The number formed by the last 2 digits are divisible by 4.	7451830<u>16</u>
5	The last digit is 0 or 5.	3629673<u>0</u>
6	The number is divisible by both 2 *and* 3.	22222<u>2</u> 2+2+2+2+2+2 = 12
8	The number formed by the last 3 digits are divisible by 8.	275287<u>128</u>
9	Sum of digit is a multiple of 9.	333666 3+3+3+6+6+6 = 27
10	The last digit is 0.	84684671<u>0</u>

6. Commutative property

Addition and multiplication operations have commutative property, which means changing the order of the numbers does not change the result.

Example:

$$\begin{aligned} & 15 \times 7 \times 4 \times 10 \\ &= 105 \times 4 \times 10 \\ &= 420 \times 10 \\ &= 4200 \end{aligned} \qquad \begin{aligned} & 15 \times 4 \times 7 \times 10 \\ &= 60 \times 7 \times 10 \\ &= 420 \times 10 \\ &= 4200 \end{aligned}$$

$$\begin{aligned} & 119 + 45 + 31 + 5 \\ &= 164 + 31 + 5 \\ &= 195 + 5 \\ &= 200 \end{aligned} \qquad \begin{aligned} & 119 + 31 + 45 + 5 \\ &= 150 + 45 + 5 \\ &= 195 + 5 \\ &= 200 \end{aligned}$$

However, subtraction and division do not have this property, making them noncommutative.

$14 - 5 \neq 5 - 14$ $\qquad$ $27 \div 9 \neq 9 \div 27$

7. "from" vs. "between"

Students commonly encounter questions such as the followings:

a) How many numbers are there *from* 3 to 11?
b) How many numbers are there *between* 3 and 11?
The two questions do not have the same answers.

The word "*from*" denotes we include all of the number mention; on the other hand, "*between*" denotes that we are only interested in the numbers after 3 and before 11.

3 4 5 6 7 8 9 10 11

From 3 to 11

3 4 5 6 7 8 9 10 11

Between 3 and 11

8. LCM and GCF

a) *Least common multiples (LCM).*

It can also be called the *lowest* common multiple.

Example: find the LCM of 10 and 12.

Below are the lists of the 10 and 12's multiples:

10: 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, …

12: 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, …

The word "common" in "least common multiples" refers to the multiples that overlaps in the two rows:

10: 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130

12: 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132

Both 60 and 120 are the common multiples for 10 and 12; there are in fact infinite numbers of common multiples for any pair of numbers.

However, the word "least" in "least common multiples" refers to the first (or smallest) multiples that overlaps in the two rows.

The LCM for 10 and 12 is therefore 60.

Formally, it is written as: $lcm\ (10, 12) = 60$

b) *Greatest common factor (GCF)*

It can also be called the greatest common *divisor* (GCD) in some textbook or contests.

Example: find the GCF of 12 and 30.

Below are the lists of 12 and 30's factors:

12: 1, 2, 3, 4, 6, 12

30: 1, 2, 3, 5, 6, 10, 15, 30

The common factors (divisors) are:

12: 1, 2, 3, 4, 6, 12

30: 1, 2, 3, 5, 6, 10, 15, 30

The word "greatest" in the "greatest common factor" refers to the largest of the common factors, which is 6 in this example.

$$gcf(12, 30) = 6$$

9. Ladder method

Students might have noticed that it is a length process to find LCM or GCF by listing the numbers' factors and multiples; ladder method is an efficient way to shorten that process.

Fun fact: it is called ladder method because of the ladder-shaped lines that are drawn under the numbers; the lines are staggered on top of each other like steps on a ladder.

Example: Find the GCF and LCM of 60 and 90.

$$\begin{array}{r|cc} 3 & 60 & 90 \\ \hline 5 & 20 & 30 \\ \hline 2 & 4 & 6 \\ \hline & 2 & 3 \end{array}$$

Write down any common factor of 60 and 90 on the left side (except 1).
3 was chosen; divide 60 and 90 by 3 and put their quotients below.
Continue the process until there aren't any common factors anymore.
The ladder process is finished!

GCF: multiply all of the numbers written on the left side. If there is only one number on the left side, that number itself would be the GCF.

$$\begin{array}{r|cc} 3 & 60 & 90 \\ \hline 5 & 20 & 30 \\ \hline 2 & 4 & 6 \\ \hline & 2 & 3 \end{array}$$

$3 \times 5 \times 2 = 30$
The GCF of 60 and 90 is therefore 30.

LCM: multiply all of the numbers written on the left side *and* on the bottom row of the ladder.

3	60	90
5	20	30
2	4	6
	2	3

$3 \times 5 \times 2 \times 2 \times 3 = 180$
The LCM of 60 and 90 is therefore 180.

Can ladder method be used to find the LCM and GCF for 3 numbers?

Answer: Yes. But the steps are slightly different.

Example: Find the GCF and LCM of 60, 90, and 48

3	60	90	48
2	20	30	16
	10	15	8

Same processes as before. Find a common factor for all 3 numbers. Write the quotients on the next ladder and continue the process.

The process for GCF is finished when there isn't a common factor that can divide into all 3 numbers given.

The GCF is therefore $3 \times 2 = 6$

However, more steps are needed to find the LCM.

Key information: after GCF is found, we only need to write down factors on the left side when it can divide into *2* of the 3 numbers. For the one number that the factor cannot divide into, we simplify copy it down to the next row. This process is continued until we cannot even find a factor for 2 of the 3 numbers left.

3	60	90	48
2	20	30	16
5	10	15	8
2	2	3	8
	1	3	4

To find LCM, we pick up from the last step of finding GCF.

5 is a common factor of 10 and 15, but not 8.
2 and 3 are quotients from last step; 8 is simplified copied down.

In the last step, 2 could divide into 2 and 8 but not 3, so 3 was carried down to the last line.

LCM: $3 \times 2 \times 5 \times 2 \times 1 \times 3 \times 4 = 720$

Note that the GCF is calculated from multiplying all of the numbers of the left side (3, 2, 5, and 2), because GCF's steps are finished the moment there isn't a common factor for all numbers.