Calculation and Operation

1. The real number system

The real number system evolved over time by expanding the notion of what we mean by the word *number*. At first, number meant something you could count, such as the number of sheep a farmer owns, these are called the *natural numbers*, or sometimes the *counting numbers*.

1) Natural Numbers **(**or counting numbers)

Examples: 1, 2, 3, 4, 5, . . .

The use of three dots at the end of the list indicates that the list keeps going forever.

If the farmer does not have any sheep, then the number of sheep that the farmer owns is zero. We call the set of natural numbers plus the number zero the *whole numbers*.

2) Whole Numbers

Examples: $\mathbf{0}, 1, 2, 3, 4, 5...$

3) Integers

The difference between whole numbers and integers is that integers include the negative ones.

Examples: . . . $-4, -3, -2, -1, 0, 1, 2, 3, 4$

How can you have a number less than zero? In Canada, winter temperature is often in the negatives, and it is not uncommon to have -20 ℃ temperature days. Another example would be owing bank money; instead of positive balance, having debts mean having negative asset.

For every real number n, there exists its *opposite,* denoted $-n$, such that the sum of n and $-n$ is zero:

$$n + (-n) = 0$$

Note that the negative sign in front of a number is part of the symbol for that number: The symbol "–3" is one object, it stands for "negative three".

The number zero is its own opposite, and zero is considered to be neither negative nor positive.

4) Rational numbers

The next generalization that we can make is to include the idea of "partial numbers" such as fractions. If we add fractions to the set of integers, we get the set of *rational numbers*.

All numbers of the form $\frac{a}{b}$ where *a* and *b* are non-zero integers are included in the rational number set.

All integers are of course also rational numbers, since they can be regarded as a fraction with a denominator of 1.

This means that all of the previous sets of numbers (natural numbers, whole numbers, and integers) are subsets of the rational numbers.

Example: although $0.\overline{6}$ never ends, it *is* a rational number because it can be expressed as $\frac{2}{3}$.

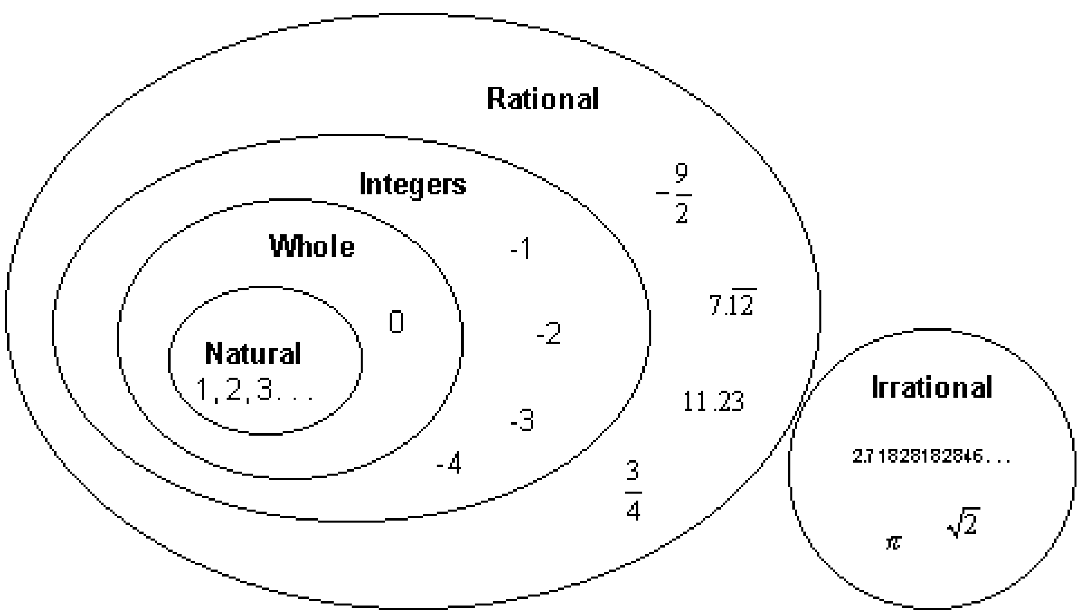
5) Irrational Numbers

Those cannot be expressed as a fraction are *irrational numbers*. In the case of a decimal, those that never terminate or repeat are considered irrational numbers.

Example: $\sqrt{2}$ and π.

6) Real numbers

When we put the irrational numbers together with the rational numbers, we finally have the complete set of real numbers. Any number that represents an amount of something, such as a weight, a volume, or the distance between two points, will be a real number. The following diagram illustrates the relationships of the sets that make up the real numbers. Are there numbers that do not belong to real numbers? Absolutely! They are called imaginary numbers and you will learn more about them in higher level classes.



2. Perfect squares

A *square number* is also called a *perfect square* sometimes, and it comes from the square of an integer; in other words, it is the product of an integer multiplying itself. For example, 9 is a square number, since it is from number 3 multiplying itself.

When a square number is prime factorized, the exponents of its prime factors are all even numbers. See below examples.

$10^2 = 100 \qquad 100 = 2^2 \cdot 5^2$

$16^2 = 256 \qquad 256 = 2^8$

$60^2 = 3600 \qquad 3600 = 2^4 \cdot 3^2 \cdot 5^2$

We can use this property to solve the below question.

Example: Find the least natural number value of k if $600k$ is a perfect square.

Step 1: prime factorization.

$600 = 2^3 \cdot 3^1 \cdot 5^2$

Step 2: to be a perfect square, the exponents of $600k$ must all be even numbers.

$2^3 \cdot 3^1 \cdot 5^2 \rightarrow 2^{\mathbf{4}} \cdot 3^{\mathbf{2}} \cdot 5^2$

Step 3: k must make up for what is missing.

$$\begin{aligned} k &= 2 \cdot 3 \\ &= 6 \end{aligned}$$

Step 4: check.

$$\begin{aligned} 600k &= 600 \cdot 6 \\ &= 3600 \end{aligned}$$

$\sqrt{3600} = 60$

3. Perfect cubes

A *perfect cube*, or *simply cube*, is a number obtained by multiplying an integer with itself three times. For example, 8 is a cube number because it is the result of number 2 multiplying itself three times.

When a cube number is prime factorized, the exponents of its prime factors are all multiples of 3. See below examples.

$10^3 = 1000$ $1000 = 2^3 \cdot 5^3$

$4^3 = 64$ $64 = 2^6$

$54^3 = 157464$ $157464 = 2^3 \cdot 3^9$

Similar to the square root example, we can use this property to solve the below question.

Example: Find the least natural number value of n if $60n$ is a cube.

Step 1: prime factorization.

$60 = 2^2 \cdot 3^1 \cdot 5^1$

Step 2: to be a perfect cube, the exponents of $60n$ must all be multiples of 3.

$2^2 \cdot 3^1 \cdot 5^1 \rightarrow 2^{\mathbf{3}} \cdot 3^{\mathbf{3}} \cdot 5^{\mathbf{3}}$

Step 3: n must make up for what is missing.

$$\begin{aligned} k &= 2^1 \cdot 3^2 \cdot 5^2 \\ &= 450 \end{aligned}$$

Step 4: check.

$$\begin{aligned} 60n &= 60 \cdot 450 \\ &= 27000 \end{aligned}$$

$\sqrt[3]{27000} = 30$

4. Order of operations

When an expression contains multiple operations, we do *not* always calculate it from left to right as there is a specific order to handle the operations. $BEDMAS$ is an acronym that summarizes the correct order of operations.

Bracket
Exponent
Division
Multiplication
Addition
Subtraction

However, division and multiplication have the same priority since they are the inverse operation of each other, and so are the addition and subtraction. Additionally, the BEDMAS rule does not include the root operation, and we will add it into the order now. Since root is the opposite operation as exponents, they share the same level of priority. We can therefore rewrite the operation order as:

1. Bracket
2. Exponents and roots
3. Division and multiplication
4. Addition and subtraction

When we encounter divisions and multiplications in the same expression, we can proceed from left to right, so are the additions and subtractions, or exponents and roots.

Example 1: Simplify $(10-2)+4\cdot 3^2$.

$(10-2)+4\cdot 3^2$	1. Bracket first.
$=8+4\cdot 3^2$	2. Exponent next.
$=8+4\cdot 9$	3. Multiplication follows
$=8+36$	4. Addition the last
$=44$	

If there are brackets inside of brackets (nested brackets), we evaluate the inner-most brackets first. The operation with the highest level of the priority for the given step is shaded in gray.

Example 2: Calculate $[72-(14-2\times 5)^3]+36\div 3^2$.

$$[72-(14-2\times 5)^3]+36\div 3^2$$
$$=[72-(14-10)^3]+36\div 3^2$$
$$=[72-(4)^3]+36\div 3^2$$
$$=[72-64]+36\div 3^2$$
$$=8+36\div 3^2$$
$$=8+36\div 9$$
$$=8+4$$
$$=12$$

Example 3: Evaluate $[-3^2+\frac{1}{5}\cdot\sqrt{225}]^2-11\cdot(-3)^3$.

$$[-3^2+\frac{1}{5}\cdot\sqrt{225}]^2-11\cdot(-3)^3$$

$$=\ [-3^2+\frac{1}{5}\cdot 15]^2-11\cdot(-3)^3$$

$$=\ [-9+\frac{1}{5}\cdot 15]^2-11\cdot(-3)^3$$

$= [-9+3]^2 - 11 \cdot (-3)^3$

$= (-6)^2 - 11 \cdot (-3)^3$

$= 36 - 11(-27)$

$= 36 + 297$

$= 333$

5. Operations with negative numbers

Let's review the rules of calculating negative numbers with a few examples.

1) Addition and subtraction

Example 1: $-8 + 6 = -2$

Example 2: $-8 - 6 = -14$

When we encounter two operation signs right next to each other, we can simplify them as:

$+- \longrightarrow -$ $\qquad$ $-- \longrightarrow +$

Example 3:

$-8 + (-6)$

$= -8 - 6$

$= -14$

Example 4:

$-8 - (-6)$

$= -8 + 6$

$= -2$

b) Multiplication and division

We can first ignore the negative signs when we multiply or divide, and decide whether we should change the result to negative depending on the rules:

1. If a number is positive and another is negative, the result is negative.
2. If both numbers are negative, the negative signs cancel out and the result is positive.

Examples:

$-10 \times 6 = -60$ $\qquad$ $-35 \div 7 = -5$
$-10 \times -6 = 60$ $\qquad$ $-35 \div -7 = 5$