Fractions

1. Fractions

A fraction can also be used to represent values in between whole numbers.

Equivalent fractions such as $\frac{1}{2}$ and $\frac{4}{8}$ have different appearances but they represent the same value, and they are called *equivalent fractions*. Equivalent fractions can be obtained by multiplying or dividing both the numerator and the denominator in a fraction by the same number.

Examples:

$$\frac{1}{2}$$

$$= \frac{1 \times 4}{2 \times 4}$$

$$= \frac{4}{8}$$

$$\frac{80}{360}$$

$$= \frac{80 \div 40}{360 \div 40}$$

$$= \frac{2}{9}$$

$\frac{1}{2}$ and $\frac{4}{8}$ are equivalent fractions; $\frac{80}{360}$ and $\frac{2}{9}$ are equivalent fractions.

As a convention, we leave the final answers in its simplest (or the most reduced) format. This rule applies to all of our homework, in class practices and the tests.

2. Reciprocals

The reciprocal of a number is one divided by that number; in another word, two numbers are each other's reciprocal if their product is 1. A and b are each other's reciprocal if:

$$a \cdot b = 1$$

For example, $\frac{2}{3}$ and $\frac{3}{2}$ is a reciprocal pair. We can say that $\frac{2}{3}$ is the reciprocal of $\frac{3}{2}$, and vice versa.

You may have noticed that we can obtain a fraction's reciprocal by reversing its numerator and denominator, but how about the reciprocal of a whole number?

We can find a whole number's reciprocal by converting it into a fraction first. Since numbers always have the same value when they are divided by 1, we are able to convert any whole numbers into a fraction out of one before finding its reciprocal, see the following example:

$20 = \frac{20}{1}$ $\qquad$ $\frac{20}{1} \times \frac{1}{20} = 1$

In the above example, 20 and $\frac{1}{20}$ are each other's reciprocals.

A negative number's reciprocal will also be a negative number; for example, -9 and $-\frac{1}{9}$ are each other's reciprocal.

3. Exponents and Roots

We follow the below rules to evaluate fractions with exponents or roots.

$(\frac{a}{b})^n = \frac{a^n}{b^n}$ $\qquad$ $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

When we encounter exponents with fractions, we can pass the exponents to both the numerator and the denominator at the same time. Root calculations follow the same rule since they are simply the inverse operation of exponents.

We recommend students to simply the fractions as soon as they see an opportunity to do so, which not only makes the calculation easier, it also saves time in the final step when we simplify the final answer.

Examples:

$(\frac{22}{363})^2$

$= (\frac{2}{33})^2 \longleftarrow$ simplify as soon as you can

$= \frac{2^2}{33^2}$

$= \frac{4}{1089}$

$\sqrt{\frac{75}{432}}$

$= \sqrt{\frac{25}{144}} \longleftarrow$ simplify as soon as you can

$= \frac{\sqrt{25}}{\sqrt{144}}$

$= \frac{5}{12}$

Note that this is different from "fractional exponents". A fractional exponent means that the exponent itself is a fraction, such as in $a^{\frac{2}{3}}$, which we will discuss later.

In most contests, we commonly leave the square root in the final answer if it cannot be reduced further, and note that sometimes, students are expected to eliminate the root in a fraction's denominator as a convention.

Example:

$\sqrt{\frac{7}{2}}$

$= \frac{\sqrt{7}}{\sqrt{2}}$ In order to eliminate the root in the denominator, we multiply $\sqrt{2}$ with both the

$= \frac{\sqrt{7}\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}$ numerator and the denominator.

$= \frac{\sqrt{7\times 2}}{2}$

$= \frac{\sqrt{14}}{2}$

4. Long calculation

Occasionally, we encounter a fairly long fraction calculation, and it is common to make calculation mistakes due to its length. We therefore suggest students to proceed this type of question in an organized manner.

In the below example, we need to start from the most inner bracket following the BEDMAS rule, which is the $1 + \frac{4}{5}$ part.

Examples:

$$1 + \cfrac{1}{1 + \cfrac{2}{1 + \cfrac{3}{1 + \cfrac{4}{5}}}}$$

Step 1: $1 + \frac{4}{5} = \frac{9}{5}$

Step 2: $3 \div \frac{9}{5} = \frac{5}{3}$

Step 3: $1 + \frac{5}{3} = \frac{8}{3}$

Step 4: $2 \div \frac{8}{3} = \frac{3}{4}$

Step 5: $1 + \frac{3}{4} = \frac{7}{4}$

Step 6: $1 \div \frac{7}{4} = \frac{4}{7}$

Step 7: $1 + \frac{4}{7} = 1\frac{4}{7}$

$\therefore \textit{The result is } 1\frac{4}{7}.$

5. Algebra

In this section, we will show a couple techniques that we can use to solve algebra questions that involves fractions.

a) We can perform cross multiplication when there is only one fraction per side of the equation.

Example: $\frac{2x-8}{4} = \frac{3x+6}{5}$

$$\frac{2x-8}{4} \times \frac{3x+6}{5}$$

$$5(2x-8) = 4(3x+6)$$

$$10x - 40 = 12x + 24$$

$$-2x = 64$$

$$x = -32$$

b) We can alternatively solve the same question by working with fractional values.

Example: $\frac{2x-8}{4} = \frac{3x+6}{5}$

$$\frac{2x}{4} - \frac{8}{4} = \frac{3x}{5} + \frac{6}{5}$$

$$\frac{1}{2}x - 2 = \frac{3}{5}x + \frac{6}{5}$$

$$\frac{1}{2}x - \frac{3}{5}x = \frac{6}{5} + 2$$

$$-\frac{1}{10}x = \frac{16}{5}$$

$$x = \frac{16}{5} \cdot -\frac{10}{1}$$

$$x = -32$$

c) Multiply the LCM of the denominators.

It is not always suitable to perform cross multiplication because there could be more than just one term per side.

Example: $\frac{5}{3x} + 10 = \frac{4}{7x}$

We can remove the fractions by multiplying the LCM of $3x$ and $7x$ with every term; their LCM is $21x$.

$$\frac{5}{3x} \cdot 21x + 10 \cdot 21x = \frac{4}{7x} \cdot 21x$$
$$35 + 210x = 12$$
$$210x = -23$$
$$x = -\frac{23}{210}$$

6. Sample questions

a) What is the value of $\frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdots \frac{18}{20} \cdot \frac{19}{21} \cdot \frac{20}{22}$?

Discussion: we should seek patterns instead of multiplying all numerators and then denominators together. The numerators are consecutive numbers from 1 to 20, and the denominators are consecutive numbers from 3 to 22; this is great!

We can cancel out the numbers that overlap between the numerators and the denominators, which are from 3 to 20. The numbers that are left are:

$$\frac{1 \cdot 2}{21 \cdot 22}$$
$$= \frac{1}{231}$$

b) Richard covered the 100km distance from home to the zoo with a speed of 50 km/h; how fast must he travel on his way back home to reach an average speed of 30km/h for the round trip?

Discussion: we can use algebra to solve the question; average speed is from dividing the total distance by the total time the journeys took.

$$Let\ x\ be\ the\ speed\ on\ the\ way\ back\ home.$$

$$\frac{100}{50} + \frac{100}{x} = \frac{200}{30}$$

$$\frac{1}{50} + \frac{1}{x} = \frac{2}{30}$$

$$\frac{1}{50} \cdot (50x) + \frac{1}{x} \cdot (50x) = \frac{2}{30} \cdot (50x)$$

$$x + 50 = \frac{10}{3}x$$

$$50 = \frac{10}{3}x - x$$

$$50 = \frac{7}{3}x$$

$$x = \frac{150}{7}$$

(or $21\frac{3}{7}$)

$$\therefore He\ must\ travel\ at\ 21\frac{3}{7}\ km/h.$$

c) Out of 20 students, what's the chance that the teachers randomly pick two students and they happen to be the tallest two kids in the class?

Discussion: if we are to call the tallest students A and B, there are two scenarios for the desired outcome to happen: AB or BA.

$P(A\ then\ B) + P(B\ then\ A)$

$= \frac{1}{20} \times \frac{1}{19} + \frac{1}{20} \times \frac{1}{19}$

$= \frac{1}{190}$

$\therefore The\ probablity\ is\ \frac{1}{190}$.