Counting and Patterns

Palindrome

Palindromes are numbers, words, or phrases that read the same forward and backward.

Examples of some palindrome names are Ava, Anna, Hannah, Aziza, and Otto; below are some examples of palindrome times:

12:21
03:30
10:01

Example question: how many 3-digit palindrome numbers are there?

The first digit cannot be 0, and there are 9 choices (digit 1-9) for this position; and there are 10 choices for the second digit (0-9), and there is only one choice for the last digit since it has to be the same as the first digit.

There are $9 \times 10 \times 1 = 90$ such numbers.

Leap year

A year is defined by the amount of time it takes for a planet to revolve around a star, and in our case on Earth, a year would be 365 days, well… not exactly.

It takes Earth around 365.24 days to complete one revolution around the Sun, and in order to account for the "extra" 0.24 of a day, one extra day is added every four years. In every four years, the accumulated one extra day $(0.24 \times 4 = 0.96)$ is included in February as February the 29th.

How do we determine whether a year is a leap year?

A year is a leap year if it is divisible by 4, most of the times.

Notice that in every four years, technically only an extra 0.96 of a day is accumulated, which is slightly short of one day, and this discrepancy is corrected by the following rules:

- The year is not a leap year if it is divisible to 100, even though it's divisible by 4.
- Among the years divisible by 100, the ones divisible by 400 are leap years.

Examples:

Leap years: 1600, 2000 (divisible by 400), 1996, 2008, 2022

Regular years: 2001, 2002, 1800, 1900, 2021

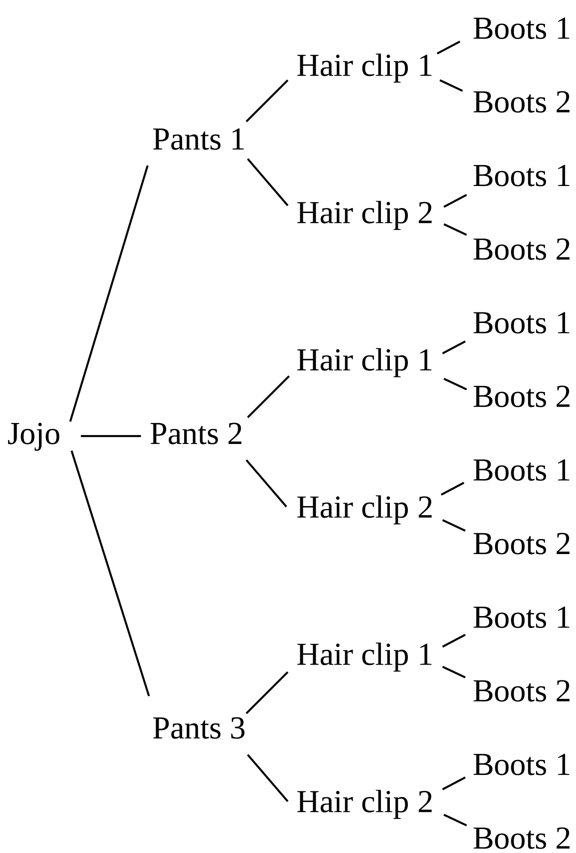
Combinations

1) The total number of combinations

In this unit, we will study the techniques of finding the number of combinations when we select one items from a set of items.

Example 1:

Jojo the border collie likes being dressed up before she goes for a walk. In her wardrobe, she has 3 different pair of pants, 2 hair clips, and 2 sets of boots. If the owner Bobbie picks one item from each category, how many combinations of outfit can there be?



In the above diagram, we can count that there are 12 possible combinations, but drawing diagrams is time consuming, is there an easier way to solve combination questions?

A second way to solve the same question:

Step 1) List the number of categories.

Pants	Hair clip	Boots

Step 2) Write down the number of *choices* per categories.

Pants	Hair clip	Boots
3	2	2

Step 3) Instead of drawing a diagram, we multiply the number of choices per category together.

Pants	Hair clip	Boots
3 ×	2 ×	2

$3 \times 2 \times 2 = 12$

$\therefore There\ are\ 12\ combinations\ in\ total\ for\ Jojo.$

Example 2:

Ethan is at a Burton store to buy snowboarding equipment. He needs a snowboard, a pair of bindings, and a pair of boots. For snowboards, there are 10 different kinds of board, there are 4 different kinds of bindings, and for boots, there is a soft, medium, and hard type. How many combinations are there for Ethan's snowboarding equipment?

Snowboard	Bindings	Boots
10 ×	4 ×	3

$10 \times 4 \times 3 = 120$

$\therefore There\ are\ 120\ combinations\ in\ total.$

Although the two examples are relatively basic questions, the principles they demonstrate form the foundation of how we would solve more complex combination questions.

2) Rearrangement questions

Example 1:

How many ways are there to rearrange the letters in word *frog*?
Discussion: the "rearrangement" questions seem to be different from the "combination" questions, but the underlying principle is shared between the two types of questions. In this case, although we don't have categories such as pants and hair clip, we have *spots* to be filled.

Step 1) In the word *frog*, there are 4 spots to be filled.

First letter	Second letter	Third letter	Fourth letter

Step 2) The tricky part is finding the number of choices per "category" (or slot).

First letter	Second letter	Third letter	Fourth letter
4 choices	3 choices	2 choices	1 choice
Any of the *f, r, o, g* can be here.	Out of the four letters *f, r, o, g*, one was taken by the previous slot.	Two out of the four letters were taken by first and second spots, so only 2 are left to be chosen.	After filling out the first 3 spots, there is only one free letter left for the final spot.

Step 3) Similar to the outfit design questions, the final step is to multiply all of the choices per category (or "spot") together.

First letter	Second letter	Third letter	Fourth letter
4 $\times$	3 $\times$	2 $\times$	1

$4 \times 3 \times 2 \times 1 = 24$

$\therefore There\ are\ 24\ ways\ to\ rearrange\ the\ word\ frog.$

Example 2:

How many ways are there to rearrange the digits in the number 3489?

The solution of this question is also 24 like in the previous example.

Similar to the "frog" rearrangement question, there are also 4 items to be arranged in this question, except that instead of letters f, r, o, g, the items are 3, 4, 8, and 9. The items in our rearrangement questions can be animals, humans, letters, numbers, or symbols, but the nature of the items do not impact the way that we calculate for their number of combinations.

The answers for the following examples are all 24 because there are 4 items to be rearranged.

a) A family of 4 wants to stand in a row to take a photo, how many ways are there to rearrange them?
b) How many ways are there to rearrange the symbols $@! ?*
c) How many ways are there to rearrange the digits in the year 1978?

3) When order matters vs. doesn't matter

Let's compare the following two questions; do you think their answers are the same?

a. In a class of 10 students, three students will be randomly selected to form a band; the first student is the singer, the second student the pianist, and the third student the guitarist. How many ways are there to assemble the band?

b. In a class of 10 students, three students will be randomly selected to form a band. How many ways are there to assemble the band?

The answers to the two above questions are different, and there are more combinations for the first scenario than for the second scenario.

The solution for scenario a:

List the number of choices for every position in the band.

Singer		Pianist		Guitarist
10	$\times$	9	$\times$	8

$10 \times 9 \times 8 = 720$

$\therefore \textit{There are } 720 \textit{ ways to assemble the band.}$

The solution for scenario b:

The first step is identical to the previous scenario.

Musician 1		Musician 2		Musician 3
10	$\times$	9	$\times$	8

$10 \times 9 \times 8 = 720$

However, since there is no specific role for each musician, their positions in the band do not matter. Consider among 10 students A, B, C, D, E, F, G, H, I, J, students B, D, G are chosen, and there are the following arrangements for the three of them:

BDG BGD DBG DGB GBD GDB

The above combinations were counted as 6 unique ways in the 72 combinations, but in reality, there is no difference among the 6 scenarios, since the order of the musicians do not matter.

For every three students chosen, there are 3! or 6 duplicated combinations. To remove the duplicates, we perform the following calculation:

$$720 \div 3! = 120$$

$\therefore \textit{There are } 120 \textit{ ways to assemble the band.}$

Students should always divide out the duplicated cases in scenarios where the order of items does not matter, such as when two teams play a match or finding the number of handshakes.

Find the last digit of a^b

Students are sometimes presented with questions such as the following:

1) Find the unit digit of 4^{1995}.
2) Find the ones digit of 2^{2026}
3) Find the ones digit of 5^{99}

Let's solve the questions by finding the pattern of these number's last digits.

1) Find the unit digit of 4^{1995}.

$4^1 = 4$
$4^2 = 16$
$4^3 = 64$
$4^4 = 256$

The last digit of 4's powers alternate between 4 and 6, and it seems like when the exponents are odd numbers, the unit digit of the result is 4, and it's 6 when the exponents are even numbers. In our example question, the exponent 1995 is an odd number, and the last digit of 4^{1995} is therefore 4.

2) Find the ones digit of 2^{2026}.

$2^1 = 2$ $2^5 = 32$
$2^2 = 4$ $2^6 = 64$
$2^3 = 8$ $2^7 = 128$
$2^4 = 16$ $2^8 = 256$

The last digit of 2's powers follow a pattern of 2, 4, 8, and 6.

$2026 \div 4 = 506R2$

In 2^{2026}, the last digit goes through 506 complete sets of 2, 4, 8, and 6, and with a remainder 2, the last digit should be the second number in the set of 2, 4, 8, and 6, which is 4.

3) Find the ones digit of 5^{99}.

$$5^1 = 5$$
$$5^2 = 25$$
$$5^3 = 125$$

The last digit of 5's powers always end in 5, making the answer of this question simply 5.