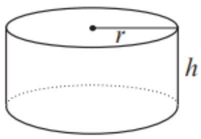
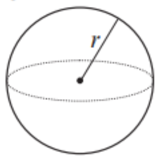
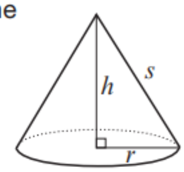
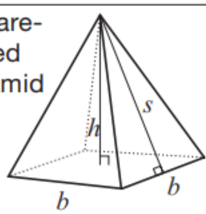
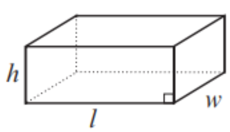
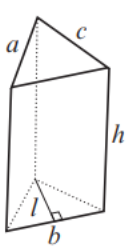
Geometry

1. 3D Geometry Formula

Geometric Figure	Surface Area	Volume
Cylinder 	$A_{\text{base}} = \pi r^2$ $A_{\text{lateral surface}} = 2\pi rh$ $A_{\text{total}} = 2A_{\text{base}} + A_{\text{lateral surface}}$ $= 2\pi r^2 + 2\pi rh$	$V = (A_{\text{base}})(\text{height})$ $V = \pi r^2 h$
Sphere 	$A = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$ or $V = \frac{4\pi r^3}{3}$
Cone 	$A_{\text{lateral surface}} = \pi rs$ $A_{\text{base}} = \pi r^2$ $A_{\text{total}} = A_{\text{lateral surface}} + A_{\text{base}}$ $= \pi rs + \pi r^2$	$V = \frac{(A_{\text{base}})(\text{height})}{3}$ $V = \frac{1}{3}\pi r^2 h$ or $V = \frac{\pi r^2 h}{3}$
Square-based pyramid 	$A_{\text{triangle}} = \frac{1}{2}bs$ $A_{\text{base}} = b^2$ $A_{\text{total}} = 4A_{\text{triangle}} + A_{\text{base}}$ $= 2bs + b^2$	$V = \frac{(A_{\text{base}})(\text{height})}{3}$ $V = \frac{1}{3}b^2 h$ or $V = \frac{b^2 h}{3}$
Rectangular prism 	$A = 2(wh + lw + lh)$	$V = (A_{\text{base}})(\text{height})$ $V = lwh$
Triangular prism 	$A_{\text{base}} = \frac{1}{2}bl$ $A_{\text{rectangles}} = ah + bh + ch$ $A_{\text{total}} = A_{\text{rectangles}} + 2A_{\text{base}}$ $= ah + bh + ch + bl$	$V = (A_{\text{base}})(\text{height})$ $V = \frac{1}{2}blh$ or $V = \frac{blh}{2}$

The above table includes the surface area and volume formula for cylinder, sphere, cone, one type of pyramid, along with two types of prisms. High school level mathematics contests rarely permit notes, and the formula of common 3D solids are considered essential for students to memorize by heart.

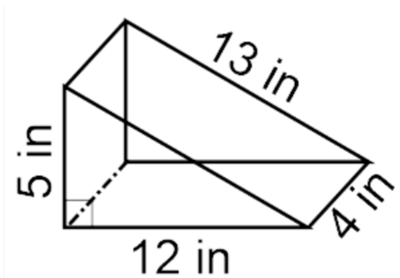
The formula sheet appears quite complex, and we encourage students to find patterns and remember formula with logic instead of rote memorization.

2. Surface Area

1) Prisms and Pyramids

3D shapes such as prisms and pyramids can be broken down into polygon faces such as squares, rectangles, or triangles. Instead of memorizing a lengthy surface area formula, we can simply observe the 3D solids, identify each face's shape, calculate the area of each face, and add the areas together.

Example:



Step 1: Identify the 3D solid.

It's a triangular prism.

Step 2: Identify its faces.

It has 5 faces: two congruent triangles and three different rectangles.

Step 3: Calculate each face's area.

- Two congruent triangles: $2\frac{5\times 12}{2} = 60$
- Left rectangle: $5 \times 4 = 20$
- Bottom rectangle: $4 \times 12 = 48$
- Top slanted rectangle: $4 \times 13 = 52$

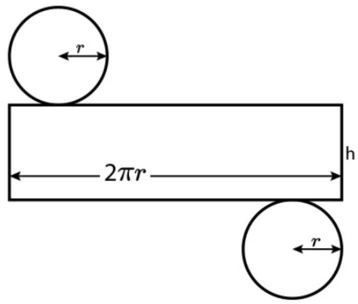
Step 4: Add the areas together.

$60 + 20 + 48 + 52 = 180$

Any prisms and pyramids' surface area can be calculated with the same logic.

2) Cylinders and cones

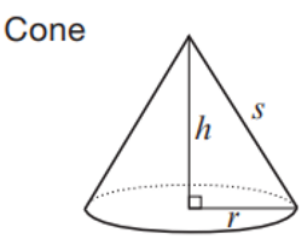
We will discuss cylinders and cones together because they both have curved surfaces.

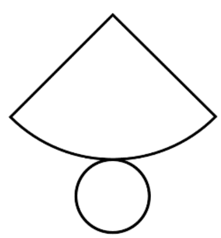


Cylinders have 3 faces: two congruent circles and a rectangular-shaped middle part. One way to visualize the rectangle part is to recall the action of pulling paper towel from the roll; even though the paper towel comes in the shape of a cylinder, the paper towel segments we tear off of the roll are rectangular-shaped.

The area of the rectangle is calculated from the base times the height. At the current level, the height of the cylinder is usually provided by the question or it can be derived with relative ease, but we still need to figure out the measurements of the base. If you observe a cylinder closely, the base of the rectangle overlaps with the edge of the circles perfectly, or in another word, the rectangle's base is the circumference of the circle, hence $2\pi r$ is the base, and the area of the rectangular part is therefore $2\pi r \cdot h$.

In cylinder's surface area formula $2\pi r^2 + 2\pi r \cdot h$, the first term corresponds to the area of the two congruent circles, and the second term indicates the area of the middle rectangular-shaped section.





The above diagrams illustrated an example of a cone and its net (broken-down) diagram. A cone is made of two faces, a circle and a "fan", which is a fraction of a circle. Students should already know that the r and h in the formula stand for radius and height, but the s might be new to some of us. The s stands for *slanted height*.

In cone's surface area formula $\pi r^2 + \pi rs$, the first term represents the area of the circle and the second terms represents the area of the side surface, or the fan-shaped part.

3. Volume

1) Prisms and cylinders

The formula for prisms' volume: $base\ area \times height$.

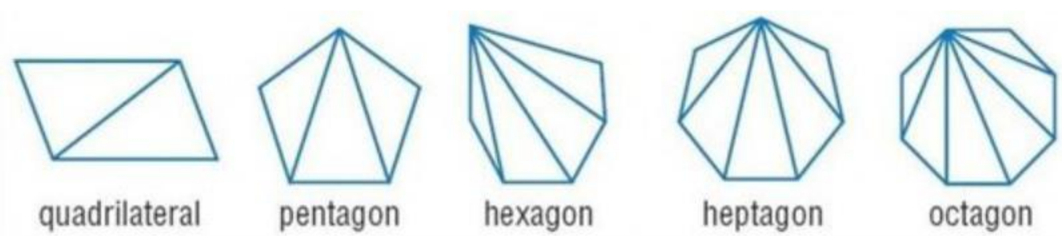
It applies to prisms such as cubes, rectangular prisms, triangular prisms, pentagonal prisms, to name a few. The challenging part therefore, is to identify the base of a prism. If the name of the shape is given, we can simply take the polygon in the prism's name as its base since the convention is to name prisms by its base's shape. For example, the triangle is the base in a triangular prism. If the name of the prism is not provided, we can find the base by looking for the following criteria:

- There are two bases.
- They are identical and parallel to each other.
- If we are to cut the prism along a plane that is in between and parallel to the two faces, the resulting cross-sections remain congruent to the base.

Although cylinder has a curved face, it shares a number of properties with prisms, and its volume can also be calculated as $base\ area \times height$. The base of a cylinder is a circle, which makes the cylinder's formula $\pi r^2 h$.

4. Sum of interior angles in polygons

We can find the sum of the interior angles of any polygon so long as we know how many sides or angles this polygon has. To illustrate how this is achieved, we can first divide polygons into triangles.



Number of sides (n)	4	5	6	7	8
Number of triangles	2	3	4	5	6

Did you find a pattern between n and the number of triangles the polygon has?

The number of triangles in a polygon: $n - 2$

The sum of angles of a triangle is $180°$, and the sum of all angles inside of a polygon is therefore: $(n-2)180°$.

Besides naming a polygon based on its number of sides, we can further classify them into regular polygons where all of its sides and angles are congruent, or irregular polygons which include all other cases.

If the given polygon is irregular, we cannot find the measurement of its individual angles without further information, and knowing the sum of all interior angles is the best we can do.

However, if the given polygon is regular, we can certainly find the measurement of an individual angle since all of them are congruent.

Example 1: find the sum of angles and the measurement of an angle in a hexagon.

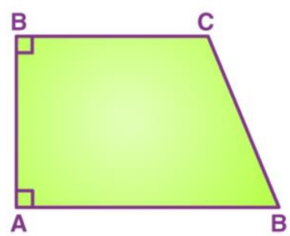
$n = 6$

$(6 - 2)180° = 720°$

$720° \div 6 = 120°$

$\therefore \mathit{The\ sum\ of\ all\ angles\ is}\ 720°\ \mathit{and\ a\ single\ angle\ measures}\ 120°.$

Example 2: find the measurement of angle C; angle B measures $78°$.



The trapezoid is a quadrilateral and it has four sides, so its sum of angles is:

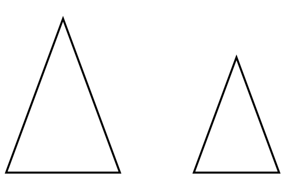
$(4 - 2)180° = 360°$

Three out of the four angles' measurements are known, and angle C can be obtained by:

$360° - 90° - 90° - 78° = 102°$

5. Similar triangles

Similar triangles have the same shape but can have different sizes, such as the below pair of triangles:



Similar triangles share the following properties:

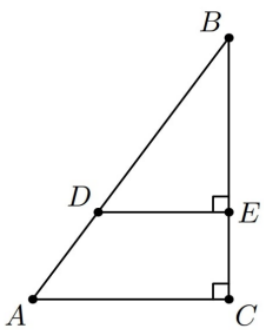
- The corresponding angles have identical measurements.
- The corresponding sides have the same ratios.

But how can we know where two triangles are similar?

Any one of the below postulates can be used to determine whether two triangles are similar.

1. $AA\sim$: Two pairs of corresponding angles are congruent.
2. $SSS\sim$: All three pairs of corresponding sides are proportional.
3. $SAS\sim$: Two pairs of corresponding sides are proportional and the corresponding angles between them are congruent.

Example: In the below shape, $BC = 8$*,* $AC = 6$*, and* $DE = 4$*, how long is* BD?



Triangles BDE and BAC are similar, because they not only share angle B, they also both contain straight (90°) angles. Postulate $AA\sim$ is used here.

Since the two legs of the larger triangle BAC are 8 and 6, the hypotenuse BA must be 10, since $6, 8,$ and 10 are Pythagorean triples.

Segment DE in triangle BDE corresponds to the segment AC in triangle BAC, and their ratio can be computed as:

$$\frac{DE}{AC} = \frac{4}{6} = \frac{2}{3}$$

Since triangles BDE and BAC are similar, their corresponding sides all share the same ratio.

$$\frac{BD}{BA} = \frac{2}{3}$$

$$\frac{BD}{10} = \frac{2}{3}$$

$$BD = \frac{20}{3}$$