Geometry

1. Unit systems

We use units to measure quantities such as length, mass, time, and volume. The appropriate units are needed to measure an entity depending on the scenarios; for example, we use micrometers to measure the thickness of certain body tissues, millimeters to measure pupil size, kilometers to measure the distance between two cities, and light years to measure the distance between celestial objects. It is also an essential skill to be able to convert between different units.

a) Metric system

The metric system is not only adopted by most countries as their official measurement systems, it is also the dominant measurement system utilized by the scientific communities. Its incorporation of the decimal system makes the conversion within the metric system very convenient.

For this course, students should be familiar with the following prefixes:

Kilo- One thousand times the base unit
Deci- One tenth the base unit
Centi- One hundredth the base unit
Milli- One thousandth the base unit

An example of base unit is metre, and below are some of the most common conversion rates for measuring length.

$1\ kilometre = 1000\ metre$ $\quad$ $1\ km = 1000\ m$

$1\ metre = 10\ decimetre$ $\quad$ $1\ m = 10\ dm$

$1\ metre = 100\ centimetre$ $\quad$ $1\ m = 100\ cm$

$1\ metre = 1000\ millimetre$ $\quad$ $1\ m = 1000\ mm$

b) Imperial system

The imperial system was first developed in the UK in the 19th century and although it is no longer the official unit system in the UK, it is nowadays still used by people in a few countries, such as the US and Liberia.

Interestingly, in Canada, although the metric system is the official measurement system, people still use imperial units in daily life; for example, we describe a person’s height with feet and inches, we measure land sizes in acres, and we sometimes measure pressure in pound-per-square-inch.

Below are the common imperial unit conversion rates that students should memorize:

Length Conversion

$1\ foot = 12\ inches$ $\quad$ $1ft = 12in$ $\quad$ $1' = 12"$

$1\ yard = 3\ feet$ $\quad$ $1\ yd = 3ft$

$1\ mile = 1760\ \ yards$ $\quad$ $1\ mi = 1760\ yd$

Mass Conversion

$1\ pound = 16\ ounces$ $\quad$ $1lb = 16\ oz.$

2. 2D and 3D unit conversions

a) 2D unit conversion

We now know that $1m = 100cm$, but does $1m^2 = 100cm^2$?

Let's solve this mystery by examining the following question.

There is a rectangle with length 4m and width 3m; what is its area in square centimetres?

Solution 1:

$4m \times 3m = 12m^2$

$12m^2 = 1200cm^2$

$\therefore The\ area\ is\ 1200cm^2.$

Solution 2:

$4m = 400cm \qquad 3m = 300cm$

$400 \times 300 = 120000cm^2$

$\therefore The\ area\ is\ 120000cm^2.$

Why do the answers differ? Which solution is correct?

Solution 2 is the correct one! Solution 1 is wrong because $1m^2 \neq 100cm^2$.

If we wish to convert the units after calculating the area, the new conversion rate can be calculated as:

$1m \times 1m = 100cm \times 100cm$

$1m^2 = 10000cm^2$

Whenever we are to convert a 2D unit, the conversion rate would be the square of the original rate.

b) 3D unit conversion

Similarly, we know that $1m^3$ is probably not the same as $100cm^2$, but what is the actual conversion rate in this case?

$1m \times 1m \times 1m = 100cm \times 100cm \times 100cm$

$1m^3 = 1000000cm^2$

$= 10^6 cm^2$

To convert any 3D units, the new conversion rate is the original rate to the power of 3.

For examples:

$1m = 10^1 dm$ $\quad (1m)^3 = (10^1 dm)^3 = 10^3 dm^3$

$1m = 10^2 cm$ $\quad (1m)^3 = (10^2 cm)^3 = 10^6 cm^3$

$1m = 10^3 mm$ $\quad (1m)^3 = (10^3 mm)^3 = 10^9 mm^3$

Example: In cm^3, what is the volume of a cube with side length of 4m?

$4^3 = 64m^3$

$64m^3 = 64 \times 10^6\ cm^3$

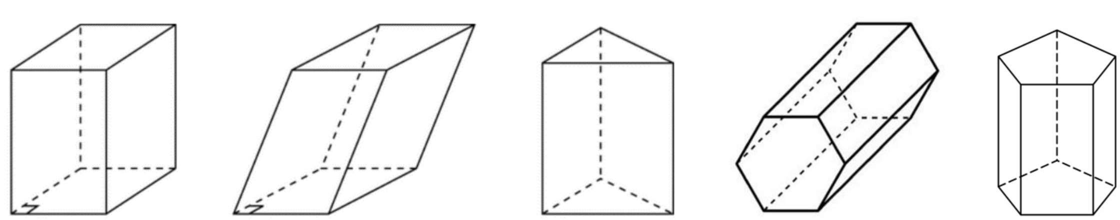
$= 64000000\ cm^3 \quad or \quad 6.4 \times 10^7 cm^3$

3. 3D geometry formulas

a) Prism

Prisms are composed of flat polygonal faces with straight edges and sharp corners. Prisms have a pair of bases, which are translated copies of each other, and when a cross-section is made parallel to the base, the cross-section is congruent to the base.

Prisms are named after their bases; below are some examples of prisms:



$$Prism's\ volume = base\ area \times height$$

In the case of a rectangular prism, the base is a rectangle which has an area of lw, and the rectangle's volume is hence lwh.

In the case of a cube, the base is a square with an area of s^2 and the height is s, which makes its volume formula s^3.

To calculate the surface area of a prism, we can add up the area of all of its faces.

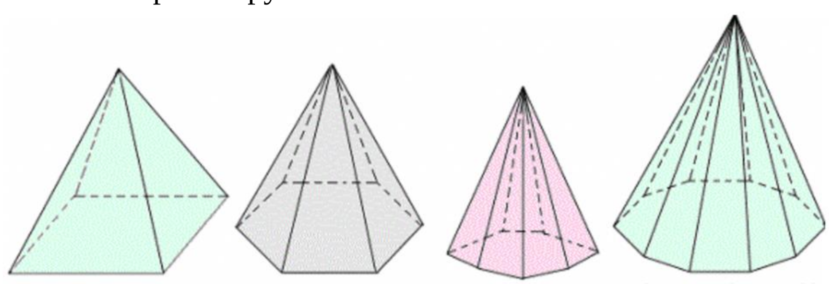
b) Pyramid

Pyramids are similar to prisms: both are made of flat polygonal faces, straight lines, and sharp corners.

The main differences:

1. pyramid has only one base, whereas prisms possess two.
2. pyramids' side faces converge to a single point called *apex*, and prisms have a flat top.
3. pyramids' sides (or *lateral faces*) are triangles; pyramids' sides are parallelograms.

Some examples of pyramids:



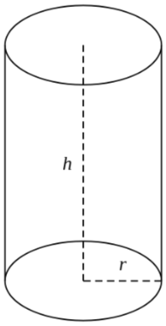
The base of the pyramid determines their names.

$$Pyramid's\ volume = \frac{1}{3} \times base\ area \times height$$

In terms of the surface area of a pyramid, we can simply add up the area of all of its faces.

c) Cylinder

A cylinder is made of 2 congruent circles and a curved side face that connect the two circles; to find its volume and surface area, all we need is its radius (or diameter) and height.

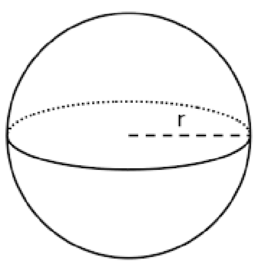


$$Cylinder's\ volume = base\ area\ \times height$$
$$= circle's\ area \times height$$
$$= \pi r^2 \cdot h$$

$$Cylinder's\ surface\ area = two\ congruent\ circles'\ area + side\ face\ area$$
$$= 2\pi r^2 + 2\pi rh$$

d) Sphere

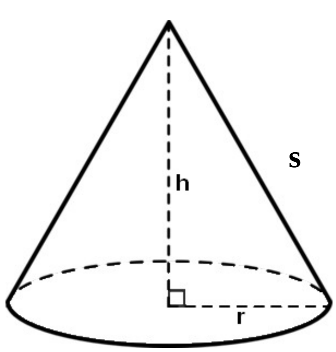
Sphere is a 3D solid; the distance between the centre and any of its surface point is always the radius.



$$Sphere's\ volume = \frac{4}{3}\pi r^3$$
$$Sphere's\ surface\ area = 4\pi r^2$$

e) Cone

Similar to a cylinder, a cone also has a circular base, but its base tapers smoothly into a singular point called apex. Birthday hats and traffic cones are examples of a cone.

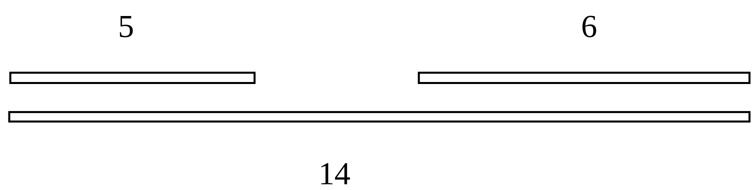


$$Cone's\ volume = \frac{1}{3} \times base\ area\ \times height$$
$$= \frac{1}{3} \cdot \pi r^2 \cdot h$$
$$Cone's\ surface\ area = base\ circle + side\ face\ area$$
$$= \pi r^2 + \pi rs \quad or \quad \pi r^2 + \pi r\sqrt{r^2 + h^2}$$

4. Constraints on triangle’s sides lengths

(Note: this discussion applies to all triangles and it is not to be confused with the Pythagorean theorem which only applies to right triangles.)

Triangles are polygons made of three sides; however, can three sides of any lengths such as 5, 6, and 14 make a triangle?



If we put the two shorter sides on top of the longest side, notice that there is the shorter two sides cannot connect to form a closed shape because they were too short; the two shorter sides can only add up to 11. Therefore, it is impossible to form a triangle with sides lengths of 5, 6, and 14.

In order to make a triangle, the sum of two sides must be larger than the third side.

$$\boxed{sum\ of\ two\ sides > third\ side}$$

Sample question: A triangle is made of integer lengths; two of the sides are 5 and 8, how many possible values are there for the missing side c?

Solution:

$5 + 8 > c$

$13 > c$

Do not stop the question here; we have to make sure that the sum of 5 $and\ c$ is greater than 8.

$5 + c > 8$

$c > 3$

To summarize, c must be an integer that is between 3 and 13.

$13 - 3 - 1 = 9$

$\therefore There\ are\ nine\ possible\ values\ for\ c.$