Algebra

1. Math format review

a) Sign omission

We do not use " × " or " ÷ " signs in algebra. Fraction replaces division sign; brackets or a dot can be used to replace multiplication sign. In the case of multiplying with a variable, we can simply write them right next to each other to show multiplication.

b) Show variable(s) in every step

The variable needs to be present in every step; it can be on the left or the right side.

c) Using equal signs properly

There should only be one equal sign per row, and the equal sign should be in between 2 expressions.

d) Final step

In the final step, the variable should be on the left side of the equation.

2. Algebraic identities

The algebraic identities are those can be satisfied with any values of the variables. They are important formulas in math as they form the foundation of algebra and are very helpful solving numerous math problems. In this section, we will describe two of the identities.

$(a+b)^2 = a^2 + 2ab + b^2$

Proof:

$(a+b)^2$

$= (a+b)(a+b)$

$= a(a+b) + b(a+b)$

$= aa + ab + ba + bb$

$= a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

Proof:

$(a-b)^2$

$= (a-b)(a-b)$

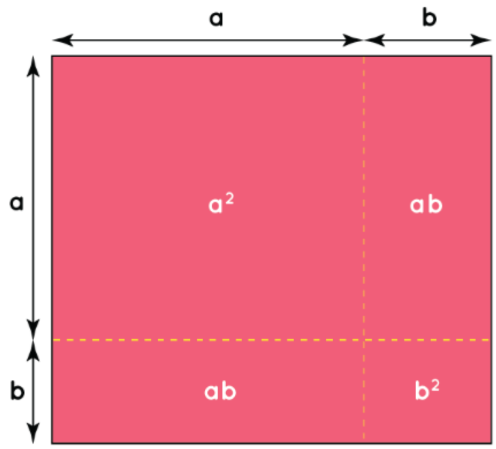
$= a(a-b) - b(a-b)$

$= aa - ab - ba + bb$

$= a^2 - ab - ab + b^2$

$= a^2 - 2ab + b^2$

Visual proof for $(a+b)^2 = a^2 + 2ab + b^2$:

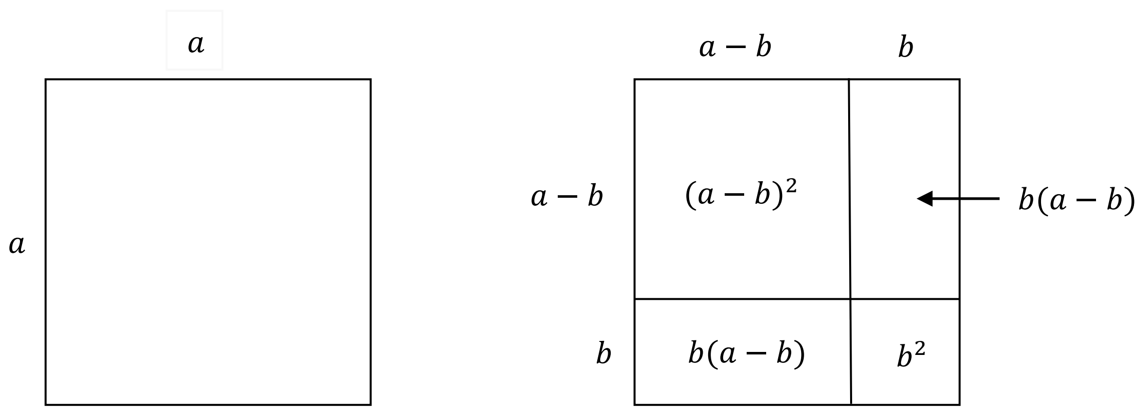


There are two ways to find the total area of the larger square:

Method 1: Side length times side length $(a+b)^2$
Method 2: Add up the four subparts $a^2 + ab + ab + b^2$ which is also $a^2 + 2ab + b^2$

It visually shows that $(a+b)^2 = a^2 + 2ab + b^2$.

Visual proof for $(a-b)^2 = a^2 - 2ab + b^2$:



Area $(a-b)^2$ can be obtained by subtracting the other three quadrilaterals from the large square's area a^2 as following:

$$a^2 - b(a-b) - b(a-b) - b^2$$

$$= a^2 - ba + b^2 - ba + b^2 - b^2$$

$$= a^2 - 2ba + b^2$$

$= a^2 - 2ab + b^2$

Let's see some examples to better understand the application of these two algebraic identities.

Example 1: Simplify the expression $(4c + 5d)^2$

$$(a + b)^2 \quad = \quad a^2 \quad + \quad 2ab \quad + \quad b^2$$

$$\downarrow \qquad \downarrow$$

$$(4c + 5d)^2 = (4c)^2 + 2(4c)(5d) + (5d)^2$$

$$= 16c^2 + 40cd + 25d^2$$

Example 2: Simplify the expression $(9g - 15)^2$

$$(a - b)^2 \quad = \quad a^2 \quad - \quad 2ab \quad + \quad b^2$$

$$\downarrow \qquad \downarrow$$

$$(9h - 15)^2 = (9g)^2 - 2(9g)(15) + (15)^2$$

$$= 81g^2 - 270g + 225$$

Example 3: Given that $h^2 + j^2 = 470$, and $hj = 35$, find the value of $h - j$.

$(h - j)^2 = h^2 - 2hj + j^2$

$(h - j)^2 = h^2 + j^2 - 2hj$

$(h - j)^2 = 470 - 2(35)$

$(h - j)^2 = 400$

$h - j = 20$

Example 4: Given that $a + b = 10$, and $ab = 16$, find the value of $a^2 + b^2$.

$a + b = 10$

$(a + b)^2 = 100$

$(a + b)^2 = a^2 + 2ab + b^2$

$100 = a^2 + 2(16) + b^2$

$a^2 + b^2 = 68$

3. Handling negative signs in algebra

When we open bracket with negative sign in the front, the sign inside the bracket should be changed. Besides memorizing the rules, students should also understand the reason behind it.

$-(x + y) = -x - y$

We can think of the negative sign in front of the bracket as "-1" multiplies the terms inside the bracket, and when we open the bracket, the logic is to first multiply -1 with x and multiply -1 with $+y$, which leaves us with $-x$ and $-y$ and we can simply put the two terms together without worrying about "changing" any signs.

Another example: $-3(x - y) = -3x + 3y$

$-3\ multiplies\ by\ x$ AND $-3\ multiplies\ by - y$, which gives $-3x\ and + 3y$.

What happens if we don't change the sign in the bracket?

Consider expression $-(10 - 7)$, we know the result is simply -3, but if we open the bracket without changing sign, the next step would be $-10 - 7 = -17$ which is wrong.

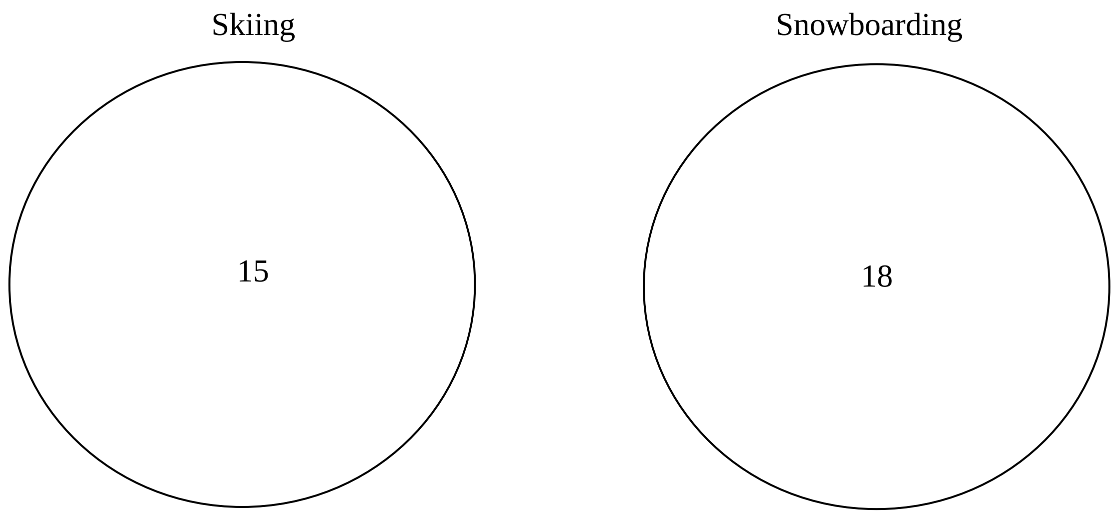
4. The "Overlapping sets" questions

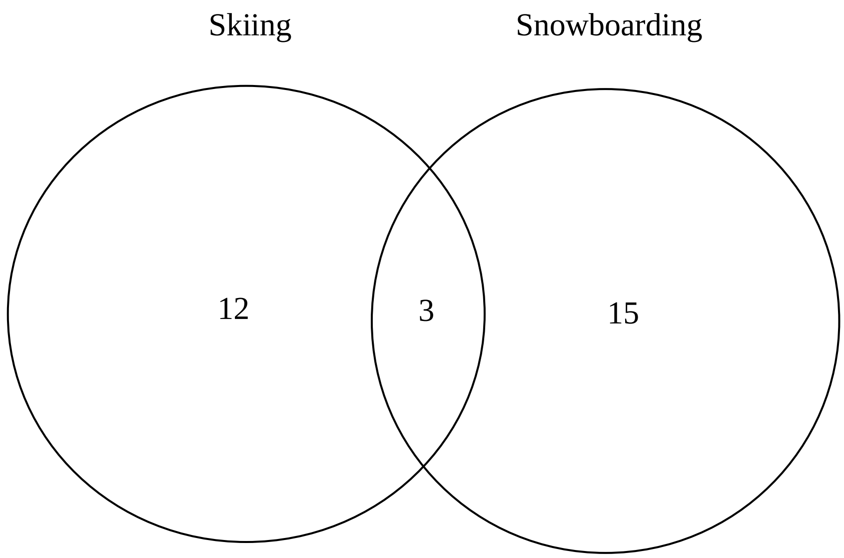
In the below example, we will demonstrate how to solve the overlapping sets question with algebra.

Sample question: As winter approaches, the school plans to take all students out for a fun day on a ski hill, but they are not sure which winter activity is more popular among their class of 30 students. When the teacher surveyed them,18 said they likes to ski and 15 said they likes to snowboard. How many students like both activities?

Discussion: the votes casted does not necessarily equates the number of students because the students can choose more than one activity. In this case, there are $18 + 15 = 33$ choices made but there are only 30 students, so $33 - 30 = 3$ students must have chosen both.

We will demonstrate this idea visually with a Venn diagram.





Out of the 15 students who chose skiing, 12 of them chose only skiing and 3 of them also chose snowboarding; out of the 18 students who chose snowboarding, 15 of them chose snowboarding only and 3 also chose skiing. The three numbers in the above Venn diagram add up exactly to 30, which is the number of students in the class.

Summary:

$$The\ sum\ of\ all\ votes\ received\ in\ both\ category - overlap = number\ of\ people$$

Example: In a group of 37 teachers, 20 voted to have the midterm test on a Monday and some teachers chose Tuesday; we know Jean, Sam, Cathy, Steven, and Evan weren't picky and they say both Monday and Tuesday work with their schedule. How many votes were received for a Tuesday midterm test?

Solution:

$$Let\ x\ be\ the\ number\ of\ votes\ received\ for\ choosing\ Tuesday?$$

$$Monday\ votes + Tuesday\ votes - overlap = number\ of\ people$$

$$20 + x - 5 = 37$$

$$x = 22$$

$$\therefore Twenty - two\ teachers\ votes\ for\ Tuesday\ midterm.$$

5. The "combined weight" questions

Sample question: Mama cat just gave birth to three very cute kittens! Too bad Ms. Lu's scale does not detect anything less than 100g so she has to weigh the kittens in pairs. She couldn't think of good names for them so she decided to call them A, B, C for now. Kitten A and B weight 200g together, kitten A and C weight 180g together, and kitten B and C weigh 220g. What is the weight of all three kittens combined?

Discussion: students do not need to calculate for the kitten's individual weight to solve the question.

$$A + B = 200$$
$$A + C = 180$$
$$B + C = 220$$

We can combine the three short equations and receive:

$$A + B + A + C + B + C = 200 + 180 + 220$$
$$2A + 2B + 2C = 600$$
$$A + B + C = 300$$

Since the goal is to calculate the combined weight of A, B, and C, we can divide both sides of the equation by 2.

$$\therefore The\ combined\ weight\ of\ the\ three\ kitten\ is\ 300g.$$